On a Group Preference Axiomatization with Cardinal Utility

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Abstract: Arrow proposed a set of conditions for a function to aggregate ordinal preferences of the members of a group, proving that it was not possible to satisfy all these assumptions simultaneously. Later, Keeney adapted these conditions and proposed a cardinal utility axiomatization for the problem of aggregating the utility functions. This work discusses the condition of nondictatorship. It proposes a stronger formulation for this condition and presents the corresponding characterization of group cardinal utility functions.

Keywords: utility functions, axiomatization, Arrow's theorem.

1 Introduction

This work focuses on the possibility of building a group utility function. There are several proposals of axioms for characterizing such a function (e.g., Harsanyi (1955), Keeney and Kirkwood (1975), Dyer and Sarin (1979), and Harvey (1999)). We focus here in particular on Keeney's (1976) group cardinal utility axiomatization, which translates to utility theory the conditions put forward by Arrow (1951) for aggregating individual rankings into a social ranking.

The contribution of this work is to revisit and reinterpret the condition of nondictatorship put forward by Arrow, in a way that makes it more consistent with common understanding of what is a dictator. A stronger condition will be proposed and the corresponding new characterizations of group utility functions are derived.

2 Arrow and Keeney's Results

Arrow (1951) addressed the problem of aggregating N individual rankings R_i ($i=1,\ldots,N$) into a group ranking R. Arrow defined that these binary relations should be weak orders. The desideratum for an aggregation method would be to obtain a social ranking R also connected and transitive. This method should satisfy five seemingly reasonable conditions: Universality, Positive association of social and individual values, Independence of irrelevant alternatives, Citizens' sovereignty, and Nondictatorship.
Keeney (1976) formulated a group cardinal utility axiomatization for certain and for uncertain alternatives. Keeney considered a set of cardinal utilities $u_i(a_j)$ concerning individuals indexed by i ($i=1,\ldots,N$) and alternatives indexed by j ($j=1,\ldots,M$) and proposed five assumptions parallel to Arrow's conditions that a group cardinal utility function $u_G = u(u_1,\ldots,u_N)$ should be consistent with. Two main results were proved. In the case of certain alternatives, u_G is consistent with those five assumptions if and only if $du/du_i \geq 0$, for $i=1,\ldots,N$, and the inequality is strict for at least two u_i's. In the case of uncertain alternatives (aggregation of von Neumann-Morgenstern expected utilities), to be consistent with the five assumptions u_G needs to be a linear combination of individual utility functions:

$$u(u_1,\ldots,u_N) = \sum_{i=1}^{N} k_i u_i \qquad (1)$$

with $k_i \geq 0$, for $i=1,\ldots,N$, and the inequality is strict for at least two k_i's. The k_i's are scaling coefficients associated with the individuals.

3 Strengthening the nondictatorship assumption

Arrow's nondictatorship condition states that "The social welfare function is not to be dictatorial", where dictatorial means "that there exists an individual *i* such that, for all *a* and b, $a\ P_i\ b$ implies *a P b* regardless of the orderings $R_1,...R_N$ of all individuals other than *i*, where *P* is the social preference relation corresponding to $R_1,...R_N$." Arrow (1951, p.30). If individual *i* is a dictator then $R = R_i$.

Keeney (1976, p. 142) formulated by analogy a nondictatorship condition (Assumption B5) stating that "There is no individual with the property that whenever he prefers alternative *a* to *b*, the group will also prefer *a* to *b* regardless of the other individual's utilities". As in the previous case, this implies that if individual *i* is a dictator then the group ranking provided by u_G coincides with the ranking implicit in u_i.

The nondictatorship conditions of Arrow and Keeney consider that a dictator is an individual so powerful that for any conceivable pair (*a*,*b*) in the space of alternatives (not necessarily the actual alternatives that a group of individuals is considering), if the dictator (an individual *i*) deems that *a* is preferred to *b*, then this yields *a P b* for the group, no matter how close $u_i(a)$ and $u_i(b)$ are.

Consider for instance that u_G follows the additive model (1), irrespective of concerning certain or uncertain alternatives. Let us also assume that (following a usual convention):

$$\forall i, u_i \in [0,1], k_i \geq 0, \text{and} \sum_{i=1}^{N} k_i = 1$$

Let us take an example in which an individual (*i*=1) has a scaling coefficient arbitrarily close to 1: k_1 = 1-ε for a small positive quantity ε (Table 1). Individual 1 is not a dictator in Keeney's (1976) sense. Indeed, no matter how small ε is, we can conceive two alternatives such that $u_G(b)>u_G(a)$, despite $u_1(b)<u_1(a)$. E.g., if $u_1(a)=c$ and $u_1(b)=c-\varepsilon$, then $u_G(b)-u_G(a)=\varepsilon^2>0$.

Table 5. Example.

	Individual 1	Individual 2	…	Individual *N*
$u_i(a)$	$u_1(a)$	0		0
$u_i(b)$	$u_1(b)$	1	…	1
k_i	1-ε	k_2		k_N

Suppose k_1=0.9990, $k_2=k_3$=0.0005. Individual 1 could argue he is not a dictator because, for instance, if $u_1(a)-u_1(b)$=0.0001, then individuals 2 and 3 could still make *b* a winner (namely if $u_2(a)=u_3(a)=0$ and $u_2(b)=u_3(b)=1$). However, this explanation would hardly convince individuals 2 and 3. A common sense reasoning that can lead to the sentiment that individual 1 is a dictator is that it is very easy for him to impose a winner, regardless of the utilities of all other individuals. For instance, if there are 4 alternatives *a*, *b*, *c*, and *d*, and individual 1 declares, e.g., $u_1(a)=1$, $u_1(b)=2/3$, $u_1(c)=1/3$, $u_1(d)=0$, then he would impose the ranking *abcd* to the group.

This reasoning acknowledges the possibility of strategic misrepresentation, but in a way that makes it more difficult to accept socially than what is usually considered in voting theory. In voting theory a method is said to be subject to strategic vote if an individual *might* get some benefit by voting strategically. Nevertheless, the fact that an individual might benefit does not guarantee he will benefit: this would require knowing the preferences of the other individuals in advance, and knowing whether these other individuals would also vote strategically.

The type of strategic misrepresentation that we can seek to prevent is arguably much more crucial to the acceptability of a group aggregation model: no individual should be able to indicate his (possibly misrepresented) preferences in a way that it guarantees that his preferences are reproduced by the group regardless. We next propose a condition to avoid such a "strategic dictator".

Condition IIW *(Immunity to imposition of a winner by an individual)*
There is no individual with the property that he can *indicate* preferences (possibly acting strategically) in a way that *guarantees* that the top alternative in the group's ranking of the alternatives coincides with the individual's preferred alternative, regardless of all other individual's preferences.

4 Axiomatization

Since the individual's utilities can be arbitrarily scaled, we will consider they lay on the [0,1] interval. Let $u_G(u_i,0_{-i})$ denote the group utility of an alternative that has utility u_i for individual i and utility zero for all other individuals. Let $u_G(u_i,1_{-i})$ denote the group utility of an alternative that has utility u_i for individual i and utility one for all other individuals. The following propositions characterize a group utility function that satisfies Keeney's assumptions plus Condition IIW:

Proposition 1. A group cardinal utility function over certain alternatives with utilities in [0,1] is consistent with Assumptions B1-B4 and IIW if and only if $du/du_i \geq 0$, for i=1,…,N, with the inequality is strict for at least two u_i's (Keeney's Theorem 1), and there is no individual i such that

$$u_G(1_i,0_{-i}) > u_G(0_i,1_{-i}).$$

Sketch of proof: if i's preferred alternative is a and he states $u_i(a)$=1 and for all $b \neq a$ states $u_i(b)$=0, then $u_G(a)>u_G(b)$, even if all other individuals $j \neq i$ state $u_j(a)$=0 and for all $b \neq a$ state $u_j(b)$=1. Keeney's Theorem 1 yields B1-B4.

Proposition 2. A group cardinal utility function over uncertain alternatives with utilities in [0,1] is consistent with Assumptions B1-B4 and IIW if and only if it has the form (1) and $k_i \geq 0$, for i=1,…,N, and there is no individual i such that

$$k_i > k_1 + \ldots + k_{i-1} + k_{i+1} + \ldots + k_N.$$

Sketch of proof: simple corollary of Prop. 1 applying (1) together with Keeney's Theorem 2.

5 Concluding notes

We introduced a concept of dictator that is more consistent with the common understanding of that term, and derived new conditions for a group utility function. An extension to address the concept of imposing a ranking is in progress.

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