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Oliveira Bastos**

Cálculo Fraccional em Escalas Temporais

Fractional Calculus on Time Scales

“Mathematical discoveries, small or great, are never born of spontaneous generation. They always presuppose a soil seeded with preliminary knowledge and well prepared by labour, both conscious and subconscious.”

Henri Poincaré



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palavras-chave

Problemas variacionais fracionais em tempo discreto, operadores fracionais discretos do tipo de Riemann—Liouville, soma por partes fraccional, equações de Euler—Lagrange, condição necessária de optimalidade do tipo de Legendre, escalas temporais, sistema de computação algébrico *Maxima*.

resumo

Introduzimos um cálculo das variações fraccional nas escalas temporais \mathbb{Z} e $(h\mathbb{Z})_a$. Estabelecemos a primeira e a segunda condição necessária de optimalidade. São dados alguns exemplos numéricos que ilustram o uso quer da nova condição de Euler—Lagrange quer da nova condição do tipo de Legendre. Introduzimos também novas definições de derivada fraccional e de integral fraccional numa escala temporal com recurso à transformada inversa generalizada de Laplace.

keywords

Fractional discrete-time variational problems, discrete analogues of Riemann—Liouville fractional-order operators, fractional formula for summation by parts, Euler—Lagrange equations, Legendre type necessary optimality condition, time scales, *Maxima* computer algebra system.

abstract

We introduce a discrete-time fractional calculus of variations on the time scales \mathbb{Z} and $(h\mathbb{Z})_a$. First and second order necessary optimality conditions are established. Some numerical examples illustrating the use of the new Euler—Lagrange and Legendre type conditions are given. We also give new definitions of fractional derivatives and integrals on time scales via the inverse generalized Laplace transform.

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Introduction

The main goal of this thesis is to develop a more general fractional calculus on time scales.

In the first year of my PhD Doctoral Programme I followed several one-semester courses in distinct fields of mathematics. One of the one-semester courses was called Research Lab where five different subjects were covered. One of those subjects was calculus of variations on time scales. Time scales theory looked to me a new and wonderful subject. As the doctoral programme PDMA Aveiro–Minho is backed up by two research units, not only Professors of the first semester but any researcher that belongs to any of these R&D units is invited to present, at the end of the first semester, research topics for PhD thesis by giving seminars to interested students. The author’s supervisor was one of the researchers that was available to be advisor if any student show interest in the theme *Fractional Calculus on Time Scales* – the title of this thesis.

The main idea was to connect in one theory two subjects that were, and still are, subject to strong research and development. Our contribute on this new area, until now, appears in part II of this thesis. On one hand we develop a discrete fractional calculus of variations for the time scale \mathbb{Z} (Chapter 3) and for the time scale $(h\mathbb{Z})_a$ (Chapter 4). On the other hand we give new definitions for differintegral fractional calculus on a time scale using a Laplace transform approach (Chapter 5).

Nowadays fractional differentiation (differentiation of an arbitrary order) plays an important role in various fields: physics (classic and quantum mechanics, thermodynamics, etc.), chemistry, biology, economics, engineering, signal and image processing, and control theory [8,60,72,73,87,94,95,100,108,110]. It is a subject as old as Calculus itself but much in progress [98]. The origin of fractional calculus goes back three centuries, when in 1695 L’Hopital asked Leibniz what should be the meaning of a derivative of order $1/2$. Leibniz’s response: “An apparent paradox, from which one day useful consequences will be drawn”.

After that episode, which most authors consider the born of fractional calculus, several famous mathematicians contributed to the development of Fractional Calculus [60, 87, 108]: Abel, Fourier, Liouville, Grünwald, Letnikov, Caputo, Riemann, Riesz, just to mention a few names. In the last decades, considerable research has been done in fractional calculus. This is particularly true in the area of the calculus of variations, which is being subject to intense investigations during the last few years [11, 25, 26, 92, 102, 103]. F. Riewe [105, 106] obtained a version of the Euler-Lagrange equations for problems of the calculus of variations with fractional derivatives, that combines the conservative and non-conservative cases. In fractional calculus of variations our main interest is the study of necessary optimality conditions for fractional problems of the calculus of variations. The study of fractional problems of the calculus of variations and respective Euler–Lagrange equations is a fairly recent issue – see [3, 4, 6, 7, 10, 14, 26, 49, 50, 55, 56, 91] and references therein – and include only the continuous case. It is well known that discrete analogues of differential equations can be very useful in applications [28, 68, 71] and that fractional Euler-Lagrange differential equations are extremely difficult to solve, being necessary to discretize them [6, 26]. Therefore, we consider pertinent to develop a fractional discrete-time theory of the calculus of variations in a different time scale than \mathbb{R} . We dedicate two chapters to that: one for the time scale \mathbb{Z} and another for the time scale $(h\mathbb{Z})_a$. Applications of fractional calculus of variations include fractional variational principles in mechanics and physics, quantization, control theory, and description of conservative, nonconservative, and constrained systems [25, 30, 31, 103]. Roughly speaking, the classical calculus of variations and optimal control are extended by substituting the usual derivatives of integer order by different kinds of fractional (non-integer) derivatives. It is important to note that the passage from the integer/classical differential calculus to the fractional one is not unique because we have at our disposal different notions of fractional derivatives. This is, as argued in [25, 102], an interesting and advantage feature of the area. Most part of investigations in the fractional variational calculus are based on the replacement of the classical derivatives by fractional derivatives in the sense of Riemann–Liouville, Caputo and Riesz [3, 10, 15, 25, 57, 93]. Independently of the chosen fractional derivatives, one obtains, when the fractional order of differentiation tends to an integer order, the usual problems and results of the calculus of variations.

A time scale is any nonempty closed subset of the real line. The theory of time scales is a fairly new area of research. It was introduced in Stefan Hilger’s 1988 Ph.D. thesis [61]

and subsequent landmark papers [62,63], as a way to unify the seemingly disparate fields of discrete dynamical systems (i.e., difference equations) and continuous dynamical systems (i.e., differential equations). His dissertation referred to such unification as “Calculus on Measure Chains” [24,75]. Today it is better known as the time scale calculus. Since the nineties of XX century, the study of dynamic equations on time scales received a lot of attention (see, e.g., [2,41,43]). In 1997, the German mathematician Martin Bohner came across time scale calculus by chance, when he took up a position at the National University of Singapore. On the way from Singapore airport, a colleague, Ravi Agarwal, mentioned that time scale calculus might be the key to the problems that Bohner was investigating at that time. After that episode, time scale calculus became one of its main areas of research. To the reader who wants a gentle overview of time scales we advise to begin with [111], that was written in a didactic way and at the same time points some possible applications (e.g. in biology). Here we are interested in the calculus of variations on time scales [85,112]. Our goal is to connect the theories of calculus of variations on time scales and fractional calculus.

This thesis is divided in two major parts. The first part has two chapters in which we provide some preliminaries on fractional calculus and time scales calculus, respectively. The second part is splitted in three chapters where we present our original work. In Chapter 3 we introduce the fractional calculus of variations on the time scale \mathbb{Z} . The main results of this chapter are a first-order necessary optimality condition (Euler-Lagrange equation) and a second-order necessary optimality condition (Legendre inequality). In Chapter 4 we introduce a fractional factorial function that allow us to define left and right fractional derivatives and to develop further the two necessary optimality conditions of Chapter 3. In Chapter 4 the time scale is $h\mathbb{Z}_a$. We believe that our results open some possible doors to research, also in the continuous case, when $h \rightarrow 0$. Chapter 5 is devoted to a new approach to define fractional derivatives and fractional integrals in an arbitrary time scale, by using the Laplace transform of time scales as support. Finally, we write our conclusions in Chapter 6 as well some future research directions.

Part I

Synthesis

Chapter 1

Fractional Calculus

“The fractional calculus is the calculus of the XXI-st century”

K. Nishimoto (1989)

The theory of discrete fractional calculus is in its infancy [20, 21, 86]. In contrast, the theory of continuous fractional calculus is much more developed [108]. The fractional discrete theory has its foundations in the pioneering work of Kuttner in 1957, where it appears the first definition of fractional order differences. In Sections 1.1 and 1.2 some definitions of fractional discrete and continuous operators are given, respectively.

1.1 Discrete fractional calculus

In 1957 [74] Kuttner defined, for any sequence of complex numbers, $\{a_n\}$, the s -th order difference as

$$\Delta^s a_n = \sum_{m=0}^{\infty} \binom{-s-1+m}{m} a_{n+m}, \quad (1.1)$$

where

$$\binom{t}{m} = \frac{t(t-1)\dots(t-m+1)}{m!}. \quad (1.2)$$

In [74] Kuttner also remarks that $\binom{t}{m}$ means 0 when m is negative and also when $t-m$ is a negative integer but t is not a negative integer. Clearly, (1.1) only makes sense when the series converges.

In 1974, Diaz and Osler [48] gave the following definition for a fractional difference of order ν :

$$\Delta^\nu f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\nu}{k} f(x + \nu - k),$$

$$\binom{\nu}{k} = \frac{\Gamma(\nu + 1)}{\Gamma(\nu - k + 1)k!} \quad (1.3)$$

where ν is any real or complex number.

The above definition uses the usual well-known gamma function which is defined by

$$\Gamma(\nu) := \int_0^{\infty} t^{\nu-1} e^{-t} dt, \quad \nu \in \mathbb{C} \setminus (-\mathbb{N}_0). \quad (1.4)$$

The gamma function was first introduced by the Swiss mathematician Leonard Euler in his goal to generalize the factorial to non integer values.

Throughout this thesis we use some of its most important properties:

$$\Gamma(1) = 1, \quad (1.5)$$

$$\Gamma(\nu + 1) = \nu \Gamma(\nu) \quad \text{for } \nu \in \mathbb{C} \setminus (-\mathbb{N}_0), \quad (1.6)$$

$$\Gamma(x + 1) = x!, \quad \text{for } x \in \mathbb{N}_0. \quad (1.7)$$

Remark 1. Using the properties of gamma function it is easily proved that formulas (1.2) and (1.3) coincide for ν an integer.

We begin by introducing some notation used throughout. Let a be an arbitrary real number and $b = a + k$ for a certain $k \in \mathbb{N}$ with $k \geq 2$. Let $\mathbb{T} = \{a, a + 1, \dots, b\}$. According with [41], we define the factorial function

$$t^{(n)} = t(t - 1)(t - 2) \dots (t - n + 1), \quad n \in \mathbb{N}$$

and $t^{(0)} = 0$.

Extending the above definition from an integer n to an arbitrary real number α , we have

$$t^{(\alpha)} = \frac{\Gamma(t + 1)}{\Gamma(t + 1 - \alpha)}, \quad (1.8)$$

where Γ is the Euler gamma function. In [86] Miller and Ross define a fractional sum of order $\nu > 0$ *via* the solution of a linear difference equation. Namely, they present it as follows:

Definition 2.

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t - \sigma(s))^{\nu-1} f(s). \quad (1.9)$$

Here f is defined for $s = a \pmod{1}$ and $\Delta^{-\nu} f$ is defined for $t = (a + \nu) \pmod{1}$.

This was done in analogy with the Riemann–Liouville fractional integral of order $\nu > 0$ (cf. formula (1.10)) which can be obtained *via* the solution of a linear differential equation [86, 87]. Some basic properties of the sum in (1.9) were obtained in [86]. Although there are other definitions of fractional difference operators, throughout this thesis, we follow mostly, the spirit of Miller and Ross, Atici and Eloe [21, 86].

1.2 Continuous fractional calculus

In the literature there are several definitions of fractional derivatives and fractional integrals (simply called differintegrals) like Riemann–Liouville, Caputo, Riesz and Hadamard. Throughout this thesis only Riemann–Liouville and Caputo definitions are used.

By historical precedent over Caputo definition, we will start with the Riemann–Liouville definition. Let $[a, b]$ be a finite interval and $\alpha \in \mathbb{R}^+$. The *left* and the *right* Riemann–Liouville fractional integrals (RLFI) of order α of a function f are defined, respectively, by:

$${}_a I_x^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t) dt, \quad x > a, \quad (1.10)$$

and

$${}_x I_b^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha-1} f(t) dt, \quad x < b, \quad (1.11)$$

where $\Gamma(\alpha)$ is the Gamma function (1.4).

Remark 3. If f is a continuous function ${}_a I_x^0 f = {}_x I_b^0 f = f$.

Remark 4. The semigroup property of Riemann–Liouville fractional operators are given by

$$({}_a I_x^\alpha {}_a I_x^\beta f)(x) = {}_a I_x^{\alpha+\beta} f(x), \quad x > a, \quad \alpha > 0, \quad \beta > 0. \quad (1.12)$$

If f is a continuous function ${}_a I_x^0 f = {}_x I_b^0 f = f$.

Let f be a function, $\alpha \in \mathbb{R}_0^+$ and $n = [\alpha] + 1$, where $[\alpha]$ means the integer part of α . The *left* and the *right* Riemann–Liouville fractional derivatives (RLFD) of order α of f

are defined, respectively, by

$${}_a D_x^\alpha f(x) := \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx} \right)^n \int_a^x (x-t)^{-\alpha+n-1} f(t) dt = \left(\frac{d}{dx} \right)^n {}_a I_x^{n-\alpha} f(x), \quad x > a \quad (1.13)$$

and

$${}_x D_b^\alpha f(x) := \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx} \right)^n \int_x^b (t-x)^{-\alpha+n-1} f(t) dt = \left(-\frac{d}{dx} \right)^n {}_x I_b^{n-\alpha} f(x), \quad x < b. \quad (1.14)$$

Let $AC([a, b])$ represent the space of absolutely continuous functions on $[a, b]$. For $n \in \mathbb{N}$ we denote by $AC^n[a, b]$ the space of functions f which have continuous derivatives up to order $n-1$ on $[a, b]$ such that $f^{(n-1)} \in AC[a, b]$. In particular, $AC^1[a, b] = AC[a, b]$.

The *left* and the *right* Caputo fractional derivatives (CFD) of order $\alpha \in \mathbb{R}_0^+$ of $f \in AC^n([a, b])$ are defined, respectively, by:

$${}_a^C D_x^\alpha f(x) := {}_a I_x^{n-\alpha} \frac{d^n}{dx^n} f(x), \quad x > a \quad (1.15)$$

and

$${}_x^C D_b^\alpha f(x) := (-1)^n {}_x I_b^{n-\alpha} \frac{d^n}{dx^n} f(x), \quad x < b, \quad (1.16)$$

where $n = [\alpha] + 1$.

Remark 5. The Riemann-Liouville approach requires the initial conditions for differential equations in terms of non-integer derivatives, which are very difficult to be interpreted from the physical point of view [90], whereas the Caputo approach uses integer-order initial conditions that are more easy to get in real world problems [89]. Caputo approach it's also preferable when we need fractional derivatives of constants to be zero. The Caputo fractional derivative is zero for a constant while the Riemann-Liouville derivative is not.

Remark 6. If $f(a) = f'(a) = \dots = f^{(n-1)}(a) = 0$, then both Riemann-Liouville and Caputo derivatives coincide. In particular, for $\alpha \in (0, 1)$ and $f(a) = 0$ one has ${}_a^C D_x^\alpha f(t) = {}_a D_x^\alpha f(t)$.

Chapter 2

Time Scales

In this chapter we recall some basic results on time scales (see Section 2.1) that we use in the sequel. In Section 2.2 we review some results of the calculus of variations on time scales.

2.1 Basic definitions

Definition 7. A *time scale* is an arbitrary nonempty closed subset of \mathbb{R} and is denoted by \mathbb{T} .

Example 8. Here we just give some examples of sets that are time scales and others that are not.

- The set \mathbb{R} is a time scale;
- The set \mathbb{Z} is a time scale;
- The Cantor set is a time scale;
- The set \mathbb{C} is not a time scale;
- The set \mathbb{Q} is not a time scale.

In applications, the quantum calculus that has, as base, the set of powers of a given number q is, in addition to classical continuous and purely discrete calculus, one of the most important time scales. Applications of this calculus appear, for example, in physics [70]

and economics [82]. The quantum derivative was introduced by Leonhard Euler and a fractional formulation of that derivative can be found in [96,97].

In this thesis attention will be given to *time scales* \mathbb{Z} , \mathbb{R} and $(h\mathbb{Z})_a = \{a, a + h, a + 2h, \dots\}$, $a \in \mathbb{R}$, $h > 0$.

A time scale of the form of a union of disjoint closed real intervals constitutes a good background for the study of population (of plants, insects, etc.) models. Such models appear, for example, when a plant population exhibits exponential growth during the months of Spring and Summer, and at the beginning of Autumn all plants die while the seeds remain in the ground. Similar examples concerning insect populations, where all the adults die before the babies are born can be found in [41,111].

The following operators of time scales theory are used, in literature and throughout this thesis, several times:

Definition 9. The mapping $\sigma : \mathbb{T} \rightarrow \mathbb{T}$, defined by $\sigma(t) = \inf \{s \in \mathbb{T} : s > t\}$ with $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(M) = M$ if \mathbb{T} has a maximum M) is called the *forward jump operator*. Accordingly, we define the *backward jump operator* $\rho : \mathbb{T} \rightarrow \mathbb{T}$ by $\rho(t) = \sup \{s \in \mathbb{T} : s < t\}$ with $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(m) = m$ if \mathbb{T} has a minimum m). The symbol \emptyset denotes the empty set.

The following classification of points is used within the theory: a point $t \in \mathbb{T}$ is called *right-dense*, *right-scattered*, *left-dense* or *left-scattered* if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, $\rho(t) < t$, respectively. A point t is called *isolated* if $\rho(t) < t < \sigma(t)$ and *dense* if $\rho(t) = t = \sigma(t)$.

Definition 10. A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called *regulated* provided its right-sided limits exist (finite) at all right-dense points in \mathbb{T} and its left-sided limits exist (finite) at all left-dense points in \mathbb{T} .

Example 11. Let $\mathbb{T} = \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{2\} \cup \left\{ 2 - \frac{1}{n} : n \in \mathbb{N} \right\}$ and define $f : \mathbb{T} \rightarrow \mathbb{R}$ by

$$f(t) := \begin{cases} 0 & \text{if } t \neq 2; \\ t & \text{if } t = 2. \end{cases} .$$

Now, let us define the sets \mathbb{T}^{κ^n} , inductively:

$$\mathbb{T}^{\kappa^1} = \mathbb{T}^{\kappa} = \{t \in \mathbb{T} : t \text{ non-maximal or left-dense}\}$$

and $\mathbb{T}^{\kappa^n} = (\mathbb{T}^{\kappa^{n-1}})^{\kappa}$, $n \geq 2$.

2.1. BASIC DEFINITIONS

Definition 12. The *forward graininess function* $\mu : \mathbb{T} \rightarrow [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$.

Remark 13. Throughout the thesis, we refer to the forward graininess function simply as the graininess function.

Definition 14. The *backward graininess function* $\nu : \mathbb{T} \rightarrow [0, \infty)$ is defined by $\nu(t) = t - \rho(t)$.

Example 15. If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = \rho(t) = t$ and $\mu(t) = \nu(t) = 0$. If $\mathbb{T} = [-4, 1] \cup \mathbb{N}$, then

$$\sigma(t) = \begin{cases} t & \text{if } t \in [-4, 1]; \\ t + 1 & \text{otherwise,} \end{cases}$$

while

$$\rho(t) = \begin{cases} t & \text{if } t \in [-4, 1]; \\ t - 1 & \text{otherwise.} \end{cases}$$

Moreover,

$$\mu(t) = \begin{cases} 0 & \text{if } t \in [-4, 1]; \\ 1 & \text{otherwise,} \end{cases}$$

and

$$\nu(t) = \begin{cases} 0 & \text{if } t \in [-4, 1]; \\ 1 & \text{otherwise.} \end{cases}$$

Example 16. If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$, $\rho(t) = t - 1$ and $\mu(t) = \nu(t) = 1$. If $\mathbb{T} = (h\mathbb{Z})_a$ then $\sigma(t) = t + h$, $\rho(t) = t - h$ and $\mu(t) = \nu(t) = h$.

For two points $a, b \in \mathbb{T}$, the time scale interval is defined by

$$[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}.$$

Throughout the thesis we will frequently write $f^{\sigma}(t) = f(\sigma(t))$ and $f^{\rho}(t) = f(\rho(t))$.

Next results are related with differentiation on time scales.

Definition 17. We say that a function $f : \mathbb{T} \rightarrow \mathbb{R}$ is Δ -differentiable at $t \in \mathbb{T}^{\kappa}$ if there is a number $f^{\Delta}(t)$ such that for all $\varepsilon > 0$ there exists a neighborhood U of t such that

$$|f^{\sigma}(t) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s| \quad \text{for all } s \in U.$$

We call $f^{\Delta}(t)$ the Δ -derivative of f at t .

The Δ -derivative of order $n \in \mathbb{N}$ of a function f is defined by $f^{\Delta^n}(t) = \left(f^{\Delta^{n-1}}(t)\right)^\Delta$, $t \in \mathbb{T}^{\kappa^n}$, provided the right-hand side of the equality exists, where $f^{\Delta^0} = f$.

Some basic properties will now be given for the Δ -derivative.

Theorem 18. [41, Theorem 1.16] Assume $f : \mathbb{T} \rightarrow \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we have the following:

1. If f is Δ -differentiable at t , then f is continuous at t .
2. If f is continuous at t and t is right-scattered, then f is Δ -differentiable at t with

$$f^\Delta(t) = \frac{f^\sigma(t) - f(t)}{\mu(t)}. \quad (2.1)$$

3. If t is right-dense, then f is Δ -differentiable at t if and only if the limit

$$\lim_{s \rightarrow t} \frac{f(s) - f(t)}{s - t}$$

exists as a finite number. In this case,

$$f^\Delta(t) = \lim_{s \rightarrow t} \frac{f(s) - f(t)}{s - t}.$$

4. If f is Δ -differentiable at t , then

$$f^\sigma(t) = f(t) + \mu(t)f^\Delta(t). \quad (2.2)$$

It is an immediate consequence of Theorem 18 that if $\mathbb{T} = \mathbb{R}$, then the Δ -derivative becomes the classical one, i.e., $f^\Delta = f'$ while if $\mathbb{T} = \mathbb{Z}$, the Δ -derivative reduces to the forward difference $f^\Delta(t) = \Delta f(t) = f(t+1) - f(t)$.

Theorem 19. [41, Theorem 1.20] Assume $f, g : \mathbb{T} \rightarrow \mathbb{R}$ are Δ -differentiable at $t \in \mathbb{T}^\kappa$. Then:

1. The sum $f + g : \mathbb{T} \rightarrow \mathbb{R}$ is Δ -differentiable at t and $(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t)$.
2. For any number $\xi \in \mathbb{R}$, $\xi f : \mathbb{T} \rightarrow \mathbb{R}$ is Δ -differentiable at t and $(\xi f)^\Delta(t) = \xi f^\Delta(t)$.
3. The product $fg : \mathbb{T} \rightarrow \mathbb{R}$ is Δ -differentiable at t and

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g^\sigma(t). \quad (2.3)$$

2.1. BASIC DEFINITIONS

Definition 20. A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called *rd-continuous* if it is continuous at right-dense points and if the left-sided limit exists at left-dense points.

Remark 21. We denote the set of all rd-continuous functions by C_{rd} or $C_{\text{rd}}(\mathbb{T})$, and the set of all Δ -differentiable functions with rd-continuous derivative by C_{rd}^1 or $C_{\text{rd}}^1(\mathbb{T})$.

Example 22. Let $\mathbb{T} = \mathbb{N}_0 \cup \left\{ 1 - \frac{1}{n} : n \in \mathbb{N} \right\}$ and define $f : \mathbb{T} \rightarrow \mathbb{R}$ by

$$f(t) := \begin{cases} 0 & \text{if } t \in \mathbb{N}; \\ t & \text{otherwise.} \end{cases}$$

It is easy to verify that f is continuous at the isolated points. So, the points that need our careful attention are the right-dense point 0 and the left-dense point 1. The right-sided limit of f at 0 exist and is finite (is equal to 0). The left-sided limit of f at 1 exist and is finite (is equal to 1). We conclude that f is rd-continuous in \mathbb{T} .

We consider now some results about integration on time scales.

Definition 23. A function $F : \mathbb{T} \rightarrow \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T}^\kappa \rightarrow \mathbb{R}$ provided $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$.

Theorem 24. [41, Theorem 1.74] *Every rd-continuous function has an antiderivative.*

Let $f : \mathbb{T}^\kappa \rightarrow \mathbb{R}$ be a rd-continuous function and let $F : \mathbb{T} \rightarrow \mathbb{R}$ be an antiderivative of f . Then, the Δ -integral is defined by $\int_a^b f(t)\Delta t = F(b) - F(a)$ for all $a, b \in \mathbb{T}$.

One can easily prove [41, Theorem 1.79] that, when $\mathbb{T} = \mathbb{R}$ then $\int_a^b f(t)\Delta t = \int_a^b f(t)dt$, being the right-hand side of the equality the usual Riemann integral, and when $[a, b] \cap \mathbb{T}$ contains only isolated points, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t=a}^{\rho(b)} \mu(t)f(t) & \text{if } a < b; \\ 0 & \text{if } a = b; \\ -\sum_{t=a}^{\rho(b)} \mu(t)f(t), & \text{if } a > b. \end{cases} \quad (2.4)$$

Remark 25. When $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = h\mathbb{Z}$, equation (2.4) holds.

The Δ -integral also satisfies

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t), \quad t \in \mathbb{T}^\kappa. \quad (2.5)$$

Theorem 26. [41, Theorem 1.77] Let $a, b, c \in \mathbb{T}$, $\xi \in \mathbb{R}$ and $f, g \in C_{rd}(\mathbb{T}^\kappa)$. Then,

$$1. \int_a^b [f(t) + g(t)] \Delta t = \int_a^b f(t) \Delta t + \int_a^b g(t) \Delta t.$$

$$2. \int_a^b (\xi f)(t) \Delta t = \xi \int_a^b f(t) \Delta t.$$

$$3. \int_a^b f(t) \Delta t = - \int_b^a f(t) \Delta t.$$

$$4. \int_a^b f(t) \Delta t = \int_a^c f(t) \Delta t + \int_c^b f(t) \Delta t.$$

$$5. \int_a^a f(t) \Delta t = 0.$$

6. If $|f(t)| \leq g(t)$ on $[a, b]_{\mathbb{T}^\kappa}$, then

$$\left| \int_a^b f(t) \Delta t \right| \leq \int_a^b g(t) \Delta t.$$

We now present the integration by parts formulas for the Δ -integral:

Lemma 27. (cf. [41, Theorem 1.77]) If $a, b \in \mathbb{T}$ and $f, g \in C_{rd}^1$, then

$$\int_a^b f^\sigma(t) g^\Delta(t) \Delta t = [(fg)(t)]_{t=a}^{t=b} - \int_a^b f^\Delta(t) g(t) \Delta t; \quad (2.6)$$

$$\int_a^b f(t) g^\Delta(t) \Delta t = [(fg)(t)]_{t=a}^{t=b} - \int_a^b f^\Delta(t) g^\sigma(t) \Delta t. \quad (2.7)$$

Remark 28. For analogous results on ∇ -integrals the reader can consult, e.g., [43].

Some more definitions and results must be presented since we will need them. We start defining functions that generalize polynomials functions in \mathbb{R} to the times scales calculus. There are, at least, two ways of doing that:

Definition 29. We define the *polynomials on time scales*, $g_k, h_k : \mathbb{T}^2 \rightarrow \mathbb{R}$ for $k \in \mathbb{N}_0$ as follows:

$$g_0(t, s) = h_0(t, s) \equiv 1 \quad \forall s, t \in \mathbb{T}$$

$$g_{k+1}(t, s) = \int_s^t g_k(\sigma(\tau), s) \Delta \tau \quad \forall s, t \in \mathbb{T},$$

$$h_{k+1}(t, s) = \int_s^t h_k(\tau, s) \Delta \tau \quad \forall s, t \in \mathbb{T}.$$

2.2. CALCULUS OF VARIATIONS

Remark 30. If we let $h_k^\Delta(t, s)$ denote, for each fixed $s \in \mathbb{T}$, the derivative of $h_k(t, s)$ with respect to t , then we have the following equalities:

1. $h_k^\Delta(t, s) = h_{k-1}(t, s)$ for $k \in \mathbb{N}$, $t \in \mathbb{T}^\kappa$;
2. $g_k^\Delta(t, s) = g_{k-1}(\sigma(t), s)$ for $k \in \mathbb{N}$, $t \in \mathbb{T}^\kappa$;
3. $g_1(t, s) = h_1(t, s) = t - s$ for all $s, t \in \mathbb{T}$.

Remark 31. It's not easy to find the explicit form of g_k and h_k for a generic time scale.

To give the reader an idea what is the formula of such polynomials for particular time scales, we give the following two examples:

Example 32. For $\mathbb{T} = \mathbb{R}$ and $k \in \mathbb{N}_0$ we have:

$$g_k(t, s) = h_k(t, s) = \frac{(t - s)^k}{k!} \quad \text{for all } s, t \in \mathbb{T}. \quad (2.8)$$

Example 33. Consider $\mathbb{T} = \overline{q^{\mathbb{Z}}} = \{q^k : k \in \mathbb{Z}\} \cup \{0\}$. For $k \in \mathbb{N}_0$ and $q > 1$ we have:

$$h_k(t, s) = \prod_{i=0}^{k-1} \frac{t - q^i s}{\sum_{j=0}^i q^j} \quad \text{for all } s, t \in \mathbb{T}. \quad (2.9)$$

2.2 Calculus of Variations

It is human nature to optimize. Typically, we try to maximize profit, minimize cost, travel to a destination following the smallest route or in the quickest time. The human propensity to optimize and the necessity to develop systematic tools for that has created an area of mathematics called optimization. Our main interest in this chapter is to introduce some concepts necessary to solve problems which involve finding extrema of an integral (functionals). The area of mathematics that deals with these problems is a branch of optimization called calculus of variations. Because this kind of calculus involve functionals it was called, earlier, functional calculus. Originally the name ‘‘calculus of variations’’ came from representing a perturbed curve using a Taylor polynomial plus some other term which was called the variation. Let us consider, for now, the following basic (but fundamental) problem: seek a function $y \in C^1[a, b]$ such that

$$\mathcal{L}[y(\cdot)] = \int_a^b L(t, y(t), y'(t)) dt \longrightarrow \min, \quad y(a) = y_a, \quad y(b) = y_b, \quad (2.10)$$

with $a, b, y_a, y_b \in \mathbb{R}$ and $L(t, u, v)$ satisfying some smoothness properties.

Remark 34. Although we write (2.10) as a minimization problem, we can formulate it as a maximization problem using the fact to minimize $\mathcal{L}[y(\cdot)]$ is the same as to maximize $-\mathcal{L}[y(\cdot)]$.

We start by giving a simple application of the calculus of variations on the real setting and then we refer to some results of the calculus of variations on time scales, which includes as special cases the classical calculus of variations ($\mathbb{T} = \mathbb{R}$) and the discrete calculus of variations ($\mathbb{T} = \mathbb{Z}$).

Example 35. Given two distinct points $A = (a, y_1)$ and $B = (b, y_2)$ in the plane \mathbb{R}^2 our task is to find the curve of shortest length connecting them. In childhood, all of us learn that the shortest path between two points is a straight line. That can be proved using, for example, the theory of calculus of variations. The formulation of this problem, in the sense of the calculus of variations, is to find the function $y(\cdot)$ that solves the following problem:

$$\mathcal{L}[y(\cdot)] = \int_a^b \sqrt{1 + (y'(t))^2} dt \longrightarrow \min, \quad y(a) = y_1, \quad y(b) = y_2. \quad (2.11)$$

In [37] Martin Bohner initiated the theory of calculus of variations on time scales. The rest of the section will be dedicated to present some results that will be necessary during the thesis. For more on the subject we refer to [51] and references therein.

Definition 36. A function f defined on $[a, b]_{\mathbb{T}} \times \mathbb{R}$ is called *continuous in the second variable, uniformly in the first variable*, if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|x_1 - x_2| < \delta$ implies $|f(t, x_1) - f(t, x_2)| < \varepsilon$ for all $t \in [a, b]_{\mathbb{T}}$.

Lemma 37 (cf. Lemma 2.2 in [37]). *Suppose that $F(x) = \int_a^b f(t, x) \Delta t$ is well defined. If f_x is continuous in x , uniformly in t , then $F'(x) = \int_a^b f_x(t, x) \Delta t$.*

Let us now extend (2.10) to a generic time-scale.

Let $a, b \in \mathbb{T}$ and $L(t, u, v) : [a, b]_{\mathbb{T}}^{\kappa} \times \mathbb{R}^2 \rightarrow \mathbb{R}$. Find a function $y \in C_{\text{rd}}^1$ such that

$$\mathcal{L}[y(\cdot)] = \int_a^b L(t, y^\sigma(t), y^\Delta(t)) \Delta t \longrightarrow \min, \quad y(a) = y_a, \quad y(b) = y_b, \quad (2.12)$$

with $y_a, y_b \in \mathbb{R}$.

Remark 38. If we fix $\mathbb{T} = \mathbb{R}$ problem (2.12) reduces to (2.10).

2.2. CALCULUS OF VARIATIONS

Remark 39. If we fix $\mathbb{T} = \mathbb{Z}$ we obtain the discrete version of (2.12). The goal is to find a function $y \in C_{\text{rd}}^1$ such that

$$\mathcal{L}[y(\cdot)] = \sum_{t=a}^{b-1} L(t, y(t+1), y^\Delta(t)) \Delta t \longrightarrow \min, \quad y(a) = y_a, \quad y(b) = y_b, \quad (2.13)$$

where $a, b \in \mathbb{Z}$ with $a < b$; $y_a, y_b \in \mathbb{R}$ and $L : \mathbb{Z} \times \mathbb{R}^2 \rightarrow \mathbb{R}$.

Definition 40. For $f \in C_{\text{rd}}^1$ we define the norm

$$\|f\| = \sup_{t \in [a, b]_{\mathbb{T}}^{\kappa}} |f^\sigma(t)| + \sup_{t \in [a, b]_{\mathbb{T}}^{\kappa}} |f^\Delta(t)|.$$

A function $\hat{y} \in C_{\text{rd}}^1$ with $\hat{y}(a) = y_a$ and $\hat{y}(b) = y_b$ is called a (weak) local minimizer for problem (2.12) provided there exists $\delta > 0$ such that $\mathcal{L}(\hat{y}) \leq \mathcal{L}(y)$ for all $y \in C_{\text{rd}}^1$ with $y(a) = y_a$ and $y(b) = y_b$ and $\|y - \hat{y}\| < \delta$.

Definition 41. A function $\eta \in C_{\text{rd}}^1$ is called an *admissible variation* provided $\eta \neq 0$ and $\eta(a) = \eta(b) = 0$.

Lemma 42 (cf. Lemma 3.4 in [37]). *Let $y, \eta \in C_{\text{rd}}^1$ be arbitrary fixed functions. Put $f(t, \varepsilon) = L(t, y^\sigma(t) + \varepsilon\eta^\sigma(t), y^\Delta(t) + \varepsilon\eta^\Delta(t))$ and $\Phi(\varepsilon) = \mathcal{L}(y + \varepsilon\eta)$, $\varepsilon \in \mathbb{R}$. If f_ε and $f_{\varepsilon\varepsilon}$ are continuous in ε , uniformly in t , then*

$$\begin{aligned} \Phi'(\varepsilon) &= \int_a^b [L_u(t, y^\sigma(t), y^\Delta(t))\eta^\sigma(t) + L_v(t, y^\sigma(t), y^\Delta(t))\eta^\Delta(t)] \Delta t, \\ \Phi''(\varepsilon) &= \int_a^b \{L_{uu}[y](t)(\eta^\sigma(t))^2 + 2\eta^\sigma(t)L_{uv}[y](t)\eta^\Delta(t) + L_{vv}[y](t)(\eta^\Delta(t))^2\} \Delta t, \end{aligned}$$

where $[y](t) = (t, y^\sigma(t), y^\Delta(t))$.

The next lemma is the time scales version of the classical Dubois–Reymond lemma.

Lemma 43 (cf. Lemma 4.1 in [37]). *Let $g \in C_{\text{rd}}([a, b]_{\mathbb{T}}^{\kappa})$. Then,*

$$\int_a^b g(t)\eta^\Delta(t)\Delta t = 0, \quad \text{for all } \eta \in C_{\text{rd}}^1([a, b]_{\mathbb{T}}) \text{ with } \eta(a) = \eta(b) = 0,$$

holds if and only if

$$g(t) = c, \quad \text{on } [a, b]_{\mathbb{T}}^{\kappa} \text{ for some } c \in \mathbb{R}.$$

The following theorem give first and second order necessary optimality conditions for problem (2.12).

Theorem 44. [37] Suppose that L satisfies the assumption of Lemma 42. If $\hat{y} \in C_{rd}^1$ is a (weak) local minimizer for problem given by (2.12), then necessarily

1. $L_v^\Delta[\hat{y}](t) = L_u[\hat{y}](t), \quad t \in [a, b]_{\mathbb{T}}^\kappa$ (time scales Euler–Lagrange equation).
2. $L_{vv}[\hat{y}](t) + \mu(t)\{2L_{uv}[\hat{y}](t) + \mu(t)L_{uu}[\hat{y}](t) + (\mu^\sigma(t))^*L_{vv}[\hat{y}](\sigma(t))\} \geq 0, \quad t \in [a, b]_{\mathbb{T}}^{\kappa^2}$
(time scales Legendre’s condition),

where $[y](t) = (t, y^\sigma(t), y^\Delta(t))$ and $\alpha^* = \frac{1}{\alpha}$ if $\alpha \in \mathbb{R} \setminus \{0\}$ and $0^* = 0$.

Using the theory of calculus of variations on time scales and considering $y_1 = 0$ and $y_2 = 1$ in Example 35, we can generalize that example to the following one.

Example 45. [37, Example 4.3] Find the solution of the problem

$$\int_a^b \sqrt{1 + (y^\Delta)^2} \Delta t \rightarrow \min, \quad y(a) = 0, \quad y(b) = 1. \quad (2.14)$$

Writing (2.14) in form of (2.12) we have

$$L(t, u, v) = \sqrt{1 + v^2}, \quad L_u(t, u, v) = 0 \quad \text{and} \quad L_v(t, u, v) = \frac{v}{1 + v^2}.$$

Suppose \hat{y} is a local minimizer of (2.14). Then, by item 1 of Theorem 44, equation

$$L_v^\Delta(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)) = 0, \quad t \in [a, b]^\kappa, \quad (2.15)$$

must hold. The last equation implies that there exist a constant $c \in \mathbb{R}$ such that

$$L_v(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)) \equiv c, \quad t \in [a, b],$$

i.e.,

$$\hat{y}^\Delta(t) = c\sqrt{1 + (\hat{y}^\Delta)^2}, \quad t \in [a, b] \quad (2.16)$$

holds. Solving equation (2.16) we obtain

$$\hat{y}(t) = \frac{t - a}{b - a} \quad \text{for all } t \in [a, b],$$

which is the equation of the straight line connecting the given points.

We refer the reader to the PhD thesis [51] for more recent results on calculus of variations on time scales. Here our main interest on the subject is just as the starting point to go further and start developing the *fractional* case on the time scale \mathbb{Z} (Chapter 3), $\mathbb{T} = (h\mathbb{Z})_a$ (Chapter 4) and possible on an arbitrary time scale (cf. Chapter 5). Before the original work of this thesis, already published in the international journals [33, 34], results on the fractional calculus of variations were restricted to the continuous case $\mathbb{T} = \mathbb{R}$ (see, e.g., [14, 55, 80]).

Part II

Original Work

Chapter 3

Fractional Variational Problems in

$$\mathbb{T} = \mathbb{Z}$$

In this chapter we introduce a discrete-time fractional calculus of variations on the time scale \mathbb{Z} . First and second order necessary optimality conditions are established. We finish the chapter with some examples illustrating the use of the new Euler–Lagrange and Legendre type conditions.

3.1 Introduction

In the last decades, considerable research has been done in fractional calculus. This is particularly true in the area of the calculus of variations, which is being subject to intense investigations during the last few years [25, 26, 33, 102, 103].

As mentioned in Section 1.1, Miller and Ross define the fractional sum of order $\nu > 0$ by

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t - \sigma(s))^{\nu-1} f(s). \quad (3.1)$$

Remark 46. This was done in analogy with the left Riemann–Liouville fractional integral of order $\nu > 0$ (cf. (1.10)),

$${}_a I_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x - s)^{\nu-1} f(s) ds,$$

which can be obtained *via* the solution of a linear differential equation [86,87]. Some basic properties of the sum in (3.1) were obtained in [86].

More recently, F. Atici and P. Eloe [20, 21] defined the fractional difference of order $\alpha > 0$, i.e., $\Delta^\alpha f(t) = \Delta^m(\Delta^{-(m-\alpha)} f(t))$ with m the least integer satisfying $m \geq \alpha$, and developed some of its properties that allow to obtain solutions of certain fractional difference equations.

To the best of the author's knowledge, the first paper with a theory for the discrete calculus of variations (using forward difference operator) was written by Tomlinson Fort [54] in 1937. Some important results of discrete calculus of variations are summarized in [71, Chap. 8]. It is well known that discrete analogues of differential equations can be very useful in applications [66, 71]. Therefore, we consider pertinent to start here a fractional discrete-time theory of the calculus of variations.

Throughout the chapter we proceed to develop the theory of *fractional difference calculus*, namely, we introduce the concept of left and right fractional sum/difference (cf. Definitions 47 and 53 below) and prove some new results related to them.

We begin, in Section 3.2, to give the definitions and results needed throughout. In Section 3.3 we present and prove the new results; in Section 3.4 we give some examples. Finally, in Section 3.5 we mention the main conclusions of the chapter, and some possible extensions and open questions. Computer code done in the Computer Algebra System Maxima is given in Appendix A.

3.2 Preliminaries

We begin by introducing some notation used throughout. Let a be an arbitrary real number and $b = k + a$ for a certain $k \in \mathbb{N}$ with $k \geq 2$. In this chapter we restrict ourselves to the time scale $\mathbb{T} = \{a, a + 1, \dots, b\}$.

Denote by \mathcal{F} the set of all real valued functions defined on \mathbb{T} . According with Example 16, we have $\sigma(t) = t + 1$, $\rho(t) = t - 1$.

The usual conventions $\sum_{t=c}^{c-1} f(t) = 0$, $c \in \mathbb{T}$, and $\prod_{i=0}^{-1} f(i) = 1$ remain valid here. As usual, the forward difference is defined by $\Delta f(t) = f^\sigma(t) - f(t)$. If we have a function f of two variables, $f(t, s)$, its partial (difference) derivatives are denoted by Δ_t and Δ_s ,

3.2. PRELIMINARIES

respectively. Recalling (1.8) we can, for arbitrary $x, y \in \mathbb{R}$, define (when it makes sense)

$$x^{(y)} = \frac{\Gamma(x+1)}{\Gamma(x+1-y)} \quad (3.2)$$

where Γ is the gamma function.

While reaching the proof of Theorem 55 we actually “find” the definition of left and right fractional sum:

Definition 47. Let $f \in \mathcal{F}$. The *left fractional sum* and the *right fractional sum* of order $\nu > 0$ are defined, respectively, as

$${}_a\Delta_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t - \sigma(s))^{\nu-1} f(s), \quad (3.3)$$

and

$${}_t\Delta_b^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=t+\nu}^b (s - \sigma(t))^{\nu-1} f(s). \quad (3.4)$$

Remark 48. The above sums (3.3) and (3.4) are defined for $t \in \{a + \nu, a + \nu + 1, \dots, b + \nu\}$ and $t \in \{a - \nu, a - \nu + 1, \dots, b - \nu\}$, respectively, while $f(t)$ is defined for $t \in \{a, a + 1, \dots, b\}$. Throughout we will write (3.3) and (3.4), respectively, in the following way:

$${}_a\Delta_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^t (t + \nu - \sigma(s))^{\nu-1} f(s), \quad t \in \mathbb{T},$$

$${}_t\Delta_b^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=t}^b (s + \nu - \sigma(t))^{\nu-1} f(s), \quad t \in \mathbb{T}.$$

Remark 49. The left fractional sum defined in (3.3) coincides with the fractional sum defined in [86] (see also (1.9)). The analogy of (3.3) and (3.4) with the Riemann–Liouville left and right fractional integrals of order $\nu > 0$ is clear:

$${}_aI_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x - s)^{\nu-1} f(s) ds,$$

$${}_xI_b^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_x^b (s - x)^{\nu-1} f(s) ds.$$

It was proved in [86] that $\lim_{\nu \rightarrow 0} {}_a\Delta_t^{-\nu} f(t) = f(t)$. We do the same for the right fractional sum using a different method. Let $\nu > 0$ be arbitrary. Then,

$$\begin{aligned} {}_t\Delta_b^{-\nu} f(t) &= \frac{1}{\Gamma(\nu)} \sum_{s=t}^b (s + \nu - \sigma(t))^{(\nu-1)} f(s) \\ &= f(t) + \frac{1}{\Gamma(\nu)} \sum_{s=\sigma(t)}^b (s + \nu - \sigma(t))^{(\nu-1)} f(s) \\ &= f(t) + \sum_{s=\sigma(t)}^b \frac{\Gamma(s + \nu - t)}{\Gamma(\nu)\Gamma(s - t + 1)} f(s) \\ &= f(t) + \sum_{s=\sigma(t)}^b \frac{\prod_{i=0}^{s-t-1} (\nu + i)}{\Gamma(s - t + 1)} f(s). \end{aligned}$$

Therefore, $\lim_{\nu \rightarrow 0} {}_t\Delta_b^{-\nu} f(t) = f(t)$. It is now natural to define

$${}_a\Delta_t^0 f(t) = {}_t\Delta_b^0 f(t) = f(t), \quad (3.5)$$

which we do here, and to write

$${}_a\Delta_t^{-\nu} f(t) = f(t) + \frac{\nu}{\Gamma(\nu + 1)} \sum_{s=a}^{t-1} (t + \nu - \sigma(s))^{(\nu-1)} f(s), \quad t \in \mathbb{T}, \quad \nu \geq 0, \quad (3.6)$$

$${}_t\Delta_b^{-\nu} f(t) = f(t) + \frac{\nu}{\Gamma(\nu + 1)} \sum_{s=\sigma(t)}^b (s + \nu - \sigma(t))^{(\nu-1)} f(s), \quad t \in \mathbb{T}, \quad \nu \geq 0.$$

The next theorem was proved in [21].

Theorem 50. [21] *Let $f \in \mathcal{F}$ and $\nu > 0$. Then, the equality*

$${}_a\Delta_t^{-\nu} \Delta f(t) = \Delta({}_a\Delta_t^{-\nu} f(t)) - \frac{(t + \nu - a)^{(\nu-1)}}{\Gamma(\nu)} f(a), \quad t \in \mathbb{T}^\kappa,$$

holds.

Remark 51. It is easy to include the case $\nu = 0$ in Theorem 50. Indeed, in view of (1.6) and (3.5), we get

$${}_a\Delta_t^{-\nu} \Delta f(t) = \Delta({}_a\Delta_t^{-\nu} f(t)) - \frac{\nu}{\Gamma(\nu + 1)} (t + \nu - a)^{(\nu-1)} f(a), \quad t \in \mathbb{T}^\kappa, \quad (3.7)$$

for all $\nu \geq 0$.

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Now, we prove the counterpart of Theorem 50 for the right fractional sum.

Theorem 52. *Let $f \in \mathcal{F}$ and $\nu \geq 0$. Then, the equality*

$${}_t\Delta_{\rho(b)}^{-\nu}\Delta f(t) = \frac{\nu}{\Gamma(\nu+1)}(b+\nu-\sigma(t))^{(\nu-1)}f(b) + \Delta({}_t\Delta_b^{-\nu}f(t)), \quad t \in \mathbb{T}^\kappa, \quad (3.8)$$

holds.

Proof. We only prove the case $\nu > 0$ as the case $\nu = 0$ is trivial (see Remark 51). We start by fixing an arbitrary $t \in \mathbb{T}^\kappa$ and prove that for all $s \in \mathbb{T}^\kappa$,

$$\Delta_s((s+\nu-\sigma(t))^{(\nu-1)}f(s)) = (\nu-1)(s+\nu-\sigma(t))^{(\nu-2)}f^\sigma(s) + (s+\nu-\sigma(t))^{(\nu-1)}\Delta f(s). \quad (3.9)$$

By definition of forward difference with respect to variable s we can write that

$$\begin{aligned} \Delta_s((s+\nu-\sigma(t))^{(\nu-1)}f(s)) &= (\sigma(s)+\nu-\sigma(t))^{(\nu-1)}f^\sigma(s) - (s+\nu-\sigma(t))^{(\nu-1)}f(s) \\ &= (s+\nu-t)^{(\nu-1)}f^\sigma(s) - (s+\nu-t-1)^{(\nu-1)}f(s) \\ &= \frac{\Gamma(s+\nu-t+1)f^\sigma(s)}{\Gamma(s-t+2)} - \frac{\Gamma(s+\nu-t)f(s)}{\Gamma(s-t+1)}. \end{aligned} \quad (3.10)$$

Applying (1.6) to the numerator and denominator of first fraction we have that (3.10) is equal to

$$\begin{aligned} &\frac{(s+\nu-t)\Gamma(s+\nu-t)f^\sigma(s)}{(s-t+1)\Gamma(s-t+1)} - \frac{\Gamma(s+\nu-t)f(s)}{\Gamma(s-t+1)} \\ &= \frac{[(s+\nu-t)f^\sigma(s) - (s-t+1)f(s)]\Gamma(s+\nu-t)}{(s-t+1)\Gamma(s-t+1)} \\ &= \frac{[(s-t+1)f^\sigma(s) + (\nu-1)f^\sigma(s) - (s-t+1)f(s)]\Gamma(s+\nu-t)}{(s-t+1)\Gamma(s-t+1)} \\ &= (\nu-1)\frac{\Gamma(s+\nu-\sigma(t)+1)}{\Gamma(s+\nu-\sigma(t)-(\nu-2)+1)}f^\sigma(s) + \frac{\Gamma(s+\nu-\sigma(t)+1)}{\Gamma(s+\nu-\sigma(t)-(\nu-1)+1)}(f^\sigma(s)-f(s)) \\ &= (\nu-1)(s+\nu-\sigma(t))^{(\nu-2)}f^\sigma(s) + (s+\nu-\sigma(t))^{(\nu-1)}\Delta f(s). \end{aligned}$$

Now, with (3.9) proven, we can state that

$$\begin{aligned}
 & \frac{1}{\Gamma(\nu)} \sum_{s=t}^{b-1} (s + \nu - \sigma(t))^{\nu-1} \Delta f(s) \\
 &= \left[\frac{(s + \nu - \sigma(t))^{\nu-1}}{\Gamma(\nu)} f(s) \right]_{s=t}^{s=b} - \frac{1}{\Gamma(\nu)} \sum_{s=t}^{b-1} (\nu - 1)(s + \nu - \sigma(t))^{\nu-2} f^\sigma(s) \\
 &= \frac{(b + \nu - \sigma(t))^{\nu-1}}{\Gamma(\nu)} f(b) - \frac{(\nu - 1)^{\nu-1}}{\Gamma(\nu)} f(t) \\
 &\quad - \frac{1}{\Gamma(\nu)} \sum_{s=t}^{b-1} (\nu - 1)(s + \nu - \sigma(t))^{\nu-2} f^\sigma(s).
 \end{aligned}$$

We now compute $\Delta({}_t\Delta_b^{-\nu} f(t))$:

$$\begin{aligned}
 \Delta({}_t\Delta_b^{-\nu} f(t)) &= \frac{1}{\Gamma(\nu)} \left[\sum_{s=\sigma(t)}^b (s + \nu - \sigma(t+1))^{\nu-1} f(s) \right. \\
 &\quad \left. - \sum_{s=t}^b (s + \nu - \sigma(t))^{\nu-1} f(s) \right] \\
 &= \frac{1}{\Gamma(\nu)} \left[\sum_{s=\sigma(t)}^b (s + \nu - \sigma(t+1))^{\nu-1} f(s) \right. \\
 &\quad \left. - \sum_{s=\sigma(t)}^b (s + \nu - \sigma(t))^{\nu-1} f(s) \right] - \frac{(\nu - 1)^{\nu-1}}{\Gamma(\nu)} f(t) \\
 &= \frac{1}{\Gamma(\nu)} \sum_{s=\sigma(t)}^b \Delta_t (s + \nu - \sigma(t))^{\nu-1} f(s) - \frac{(\nu - 1)^{\nu-1}}{\Gamma(\nu)} f(t) \\
 &= -\frac{1}{\Gamma(\nu)} \sum_{s=t}^{b-1} (\nu - 1)(s + \nu - \sigma(t))^{\nu-2} f^\sigma(s) - \frac{(\nu - 1)^{\nu-1}}{\Gamma(\nu)} f(t).
 \end{aligned}$$

Since t is arbitrary, the theorem is proved. \square

Definition 53. Let $0 < \alpha \leq 1$ and set $\mu = 1 - \alpha$. Then, the *left fractional difference* and the *right fractional difference* of order α of a function $f \in \mathcal{F}$ are defined, respectively, by

$${}_a\Delta_t^\alpha f(t) = \Delta({}_a\Delta_t^{-\mu} f(t)), \quad t \in \mathbb{T}^\kappa,$$

and

$${}_t\Delta_b^\alpha f(t) = -\Delta({}_t\Delta_b^{-\mu} f(t)), \quad t \in \mathbb{T}^\kappa.$$

3.3 Main results

Our aim is to introduce the discrete-time (in time scale $\mathbb{T} = \{a, a+1, \dots, b\}$) fractional problem of the calculus of variations and to prove corresponding necessary optimality conditions. In order to obtain an analogue of the Euler–Lagrange equation (cf. Theorem 59) we first prove a fractional formula of summation by parts. Our results give discrete analogues to the fractional Riemann–Liouville results available in the literature: Theorem 55 is the discrete analog of fractional integration by parts [105, 108]; Theorem 59 is the discrete analog of the fractional Euler–Lagrange equation of Agrawal [3, Theorem 1]; the natural boundary conditions (3.24) and (3.25) are the discrete fractional analogues of the transversality conditions in [5, 10]. However, to the best of the author’s knowledge, no counterpart to our Theorem 62 exists in the literature of continuous fractional variational problems.

3.3.1 Fractional summation by parts

The next lemma is used in the proof of Theorem 55.

Lemma 54. *Let f and h be two functions defined on \mathbb{T}^κ and g a function defined on $\mathbb{T}^\kappa \times \mathbb{T}^\kappa$. Then, the equality*

$$\sum_{\tau=a}^{b-1} f(\tau) \sum_{s=a}^{\tau-1} g(\tau, s) h(s) = \sum_{\tau=a}^{b-2} h(\tau) \sum_{s=\sigma(\tau)}^{b-1} g(s, \tau) f(s)$$

holds.

Proof. Choose $\mathbb{T} = \mathbb{Z}$ and $F(\tau, s) = f(\tau)g(\tau, s)h(s)$ in Theorem 10 of [9]. □

The next result gives a *fractional summation by parts* formula.

Theorem 55 (Fractional summation by parts). *Let f and g be real valued functions defined on \mathbb{T}^k and \mathbb{T} , respectively. Fix $0 < \alpha \leq 1$ and put $\mu = 1 - \alpha$. Then,*

$$\begin{aligned} \sum_{t=a}^{b-1} f(t)_a \Delta_t^\alpha g(t) &= f(b-1)g(b) - f(a)g(a) + \sum_{t=a}^{b-2} {}_t\Delta_{\rho(b)}^\alpha f(t) g^\sigma(t) \\ &+ \frac{\mu}{\Gamma(\mu+1)} g(a) \left(\sum_{t=a}^{b-1} (t+\mu-a)^{(\mu-1)} f(t) - \sum_{t=\sigma(a)}^{b-1} (t+\mu-\sigma(a))^{(\mu-1)} f(t) \right). \end{aligned}$$

Proof. From (3.7) we can write

$$\begin{aligned}
 \sum_{t=a}^{b-1} f(t) {}_a\Delta_t^\alpha g(t) &= \sum_{t=a}^{b-1} f(t) \Delta({}_a\Delta_t^{-\mu} g(t)) \\
 &= \sum_{t=a}^{b-1} f(t) \left[{}_a\Delta_t^{-\mu} \Delta g(t) + \frac{\mu}{\Gamma(\mu+1)} (t+\mu-a)^{(\mu-1)} g(a) \right] \\
 &= \sum_{t=a}^{b-1} f(t) {}_a\Delta_t^{-\mu} \Delta g(t) + \sum_{t=a}^{b-1} \frac{\mu}{\Gamma(\mu+1)} (t+\mu-a)^{(\mu-1)} f(t) g(a). \quad (3.11)
 \end{aligned}$$

Using (3.6) we get

$$\begin{aligned}
 &\sum_{t=a}^{b-1} f(t) {}_a\Delta_t^{-\mu} \Delta g(t) \\
 &= \sum_{t=a}^{b-1} f(t) \Delta g(t) + \frac{\mu}{\Gamma(\mu+1)} \sum_{t=a}^{b-1} f(t) \sum_{s=a}^{t-1} (t+\mu-\sigma(s))^{(\mu-1)} \Delta g(s) \\
 &= \sum_{t=a}^{b-1} f(t) \Delta g(t) + \frac{\mu}{\Gamma(\mu+1)} \sum_{t=a}^{b-2} \Delta g(t) \sum_{s=\sigma(t)}^{b-1} (s+\mu-\sigma(t))^{(\mu-1)} f(s) \\
 &= f(b-1)[g(b) - g(b-1)] + \sum_{t=a}^{b-2} \Delta g(t) {}_t\Delta_{\rho(b)}^{-\mu} f(t),
 \end{aligned}$$

where the third equality follows by Lemma 54. We proceed to develop the right hand side of the last equality as follows:

$$\begin{aligned}
 &f(b-1)[g(b) - g(b-1)] + \sum_{t=a}^{b-2} \Delta g(t) {}_t\Delta_{\rho(b)}^{-\mu} f(t) \\
 &= f(b-1)[g(b) - g(b-1)] + \left[g(t) {}_t\Delta_{\rho(b)}^{-\mu} f(t) \right]_{t=a}^{t=b-1} - \sum_{t=a}^{b-2} g^\sigma(t) \Delta({}_t\Delta_{\rho(b)}^{-\mu} f(t)) \\
 &= f(b-1)g(b) - f(a)g(a) - \frac{\mu}{\Gamma(\mu+1)} g(a) \sum_{s=\sigma(a)}^{b-1} (s+\mu-\sigma(a))^{(\mu-1)} f(s) \\
 &\quad + \sum_{t=a}^{b-2} ({}_t\Delta_{\rho(b)}^\alpha f(t)) g^\sigma(t),
 \end{aligned}$$

where the first equality follows from the usual summation by parts formula. Putting this

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into (3.11), we get:

$$\begin{aligned} \sum_{t=a}^{b-1} f(t)_a \Delta_t^\alpha g(t) &= f(b-1)g(b) - f(a)g(a) + \sum_{t=a}^{b-2} ({}_t \Delta_{\rho(b)}^\alpha f(t)) g^\sigma(t) \\ &+ \frac{g(a)\mu}{\Gamma(\mu+1)} \sum_{t=a}^{b-1} \frac{(t+\mu-a)^{(\mu-1)}}{\Gamma(\mu)} f(t) - \frac{g(a)\mu}{\Gamma(\mu+1)} \sum_{s=\sigma(a)}^{b-1} (s+\mu-\sigma(a))^{(\mu-1)} f(s). \end{aligned}$$

The theorem is proved. \square

3.3.2 Necessary optimality conditions

We begin to fix two arbitrary real numbers α and β such that $\alpha, \beta \in (0, 1]$. Further, we put $\mu = 1 - \alpha$ and $\nu = 1 - \beta$.

Let a function $L(t, u, v, w) : \mathbb{T}^\kappa \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be given. We assume that the second-order partial derivatives $L_{uu}, L_{uv}, L_{uw}, L_{vv}, L_{vw}$, and L_{ww} exist and are continuous.

Consider the functional $\mathcal{L} : \mathcal{F} \rightarrow \mathbb{R}$ defined by

$$\mathcal{L}(y(\cdot)) = \sum_{t=a}^{b-1} L(t, y^\sigma(t), {}_a \Delta_t^\alpha y(t), {}_t \Delta_b^\beta y(t)) \quad (3.12)$$

and the problem, that we denote by (P), of minimizing (3.12) subject to the boundary conditions $y(a) = A$ and $y(b) = B$ ($A, B \in \mathbb{R}$). Our aim is to derive necessary conditions of first and second order for problem (P).

Definition 56. For $f \in \mathcal{F}$ we define the norm

$$\|f\| = \max_{t \in \mathbb{T}^\kappa} |f^\sigma(t)| + \max_{t \in \mathbb{T}^\kappa} |{}_a \Delta_t^\alpha f(t)| + \max_{t \in \mathbb{T}^\kappa} |{}_t \Delta_b^\beta f(t)|.$$

A function $\tilde{y} \in \mathcal{F}$ with $\tilde{y}(a) = A$ and $\tilde{y}(b) = B$ is called a local minimizer for problem (P) provided there exists $\delta > 0$ such that $\mathcal{L}(\tilde{y}) \leq \mathcal{L}(y)$ for all $y \in \mathcal{F}$ with $y(a) = A$ and $y(b) = B$ and $\|y - \tilde{y}\| < \delta$.

Remark 57. It is easy to see that Definition 56 gives a norm in \mathcal{F} . Indeed, it is clear that $\|f\|$ is nonnegative, and for an arbitrary $f \in \mathcal{F}$ and $k \in \mathbb{R}$ we have $\|kf\| = |k|\|f\|$. The

triangle inequality is also easy to prove:

$$\begin{aligned}
 \|f + g\| &= \max_{t \in \mathbb{T}^\kappa} |f(t) + g(t)| + \max_{t \in \mathbb{T}^\kappa} |{}_a\Delta_t^\alpha(f + g)(t)| + \max_{t \in \mathbb{T}^\kappa} |{}_t\Delta_b^\alpha(f + g)(t)| \\
 &\leq \max_{t \in \mathbb{T}^\kappa} [|f(t)| + |g(t)|] + \max_{t \in \mathbb{T}^\kappa} [|{}_a\Delta_t^\alpha f(t)| + |{}_a\Delta_t^\alpha g(t)|] \\
 &\quad + \max_{t \in \mathbb{T}^\kappa} [|{}_t\Delta_b^\alpha f(t)| + |{}_t\Delta_b^\alpha g(t)|] \\
 &\leq \|f\| + \|g\|.
 \end{aligned}$$

The only possible doubt is to prove that $\|f\| = 0$ implies that $f(t) = 0$ for any $t \in \mathbb{T} = \{a, a + 1, \dots, b\}$. Suppose $\|f\| = 0$. It follows that

$$\max_{t \in \mathbb{T}^\kappa} |f^\sigma(t)| = 0, \quad (3.13)$$

$$\max_{t \in \mathbb{T}^\kappa} |{}_a\Delta_t^\alpha f(t)| = 0, \quad (3.14)$$

$$\max_{t \in \mathbb{T}^\kappa} |{}_t\Delta_b^\beta f(t)| = 0. \quad (3.15)$$

From (3.13) we conclude that $f(t) = 0$ for all $t \in \{a + 1, \dots, b\}$. It remains to prove that $f(a) = 0$. To prove this we use (3.14) (or (3.15)). Indeed, from (3.13) we can write

$$\begin{aligned}
 {}_a\Delta_t^\alpha f(t) &= \Delta \left(\frac{1}{\Gamma(1 - \alpha)} \sum_{s=a}^t (t + 1 - \alpha - \sigma(s))^{(-\alpha)} f(s) \right) \\
 &= \frac{1}{\Gamma(1 - \alpha)} \left(\sum_{s=a}^{t+1} (t + 2 - \alpha - \sigma(s))^{(-\alpha)} f(s) - \sum_{s=a}^t (t + 1 - \alpha - \sigma(s))^{(-\alpha)} f(s) \right) \\
 &= \frac{1}{\Gamma(1 - \alpha)} \left((t + 2 - \alpha - \sigma(a))^{(-\alpha)} f(a) - (t + 1 - \alpha - \sigma(a))^{(-\alpha)} f(a) \right) \\
 &= \frac{f(a)}{\Gamma(1 - \alpha)} \Delta(t + 1 - \alpha - \sigma(a))^{(-\alpha)}
 \end{aligned}$$

and since by (3.14) ${}_a\Delta_t^\alpha f(t) = 0$, one concludes that $f(a) = 0$ (because $(t + 1 - \alpha - \sigma(a))^{(-\alpha)}$ is not a constant).

Definition 58. A function $\eta \in \mathcal{F}$ is called an admissible variation for problem (P) provided $\eta \neq 0$ and $\eta(a) = \eta(b) = 0$.

The next theorem presents a first order necessary optimality condition for problem (P).

Theorem 59 (The fractional discrete-time Euler–Lagrange equation). *If $\tilde{y} \in \mathcal{F}$ is a local minimizer for problem (P), then*

$$L_u[\tilde{y}](t) + {}_t\Delta_{\rho(b)}^\alpha L_v[\tilde{y}](t) + {}_a\Delta_t^\beta L_w[\tilde{y}](t) = 0 \quad (3.16)$$

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holds for all $t \in \mathbb{T}^{\kappa^2}$, where the operator $[\cdot]$ is defined by

$$[y](s) = (s, y^\sigma(s), {}_a\Delta_s^\alpha y(s), {}_s\Delta_b^\beta y(s)).$$

Proof. Suppose that $\tilde{y}(\cdot)$ is a local minimizer of $\mathcal{L}(\cdot)$. Let $\eta(\cdot)$ be an arbitrary fixed admissible variation and define the function $\Phi : \left(-\frac{\delta}{\|\eta(\cdot)\|}, \frac{\delta}{\|\eta(\cdot)\|}\right) \rightarrow \mathbb{R}$ by

$$\Phi(\varepsilon) = \mathcal{L}(\tilde{y}(\cdot) + \varepsilon\eta(\cdot)). \quad (3.17)$$

This function has a minimum at $\varepsilon = 0$, so we must have $\Phi'(0) = 0$, i.e.,

$$\sum_{t=a}^{b-1} \left[L_u[\tilde{y}](t)\eta^\sigma(t) + L_v[\tilde{y}](t)_a\Delta_t^\alpha\eta(t) + L_w[\tilde{y}](t)_t\Delta_b^\beta\eta(t) \right] = 0,$$

which we may write, equivalently, as

$$\begin{aligned} L_u[\tilde{y}](t)\eta^\sigma(t)|_{t=\rho(b)} + \sum_{t=a}^{b-2} L_u[\tilde{y}](t)\eta^\sigma(t) \\ + \sum_{t=a}^{b-1} L_v[\tilde{y}](t)_a\Delta_t^\alpha\eta(t) + \sum_{t=a}^{b-1} L_w[\tilde{y}](t)_t\Delta_b^\beta\eta(t) = 0. \end{aligned} \quad (3.18)$$

Using Theorem 55, and the fact that $\eta(a) = \eta(b) = 0$, we get for the third term in (3.18) that

$$\sum_{t=a}^{b-1} L_v[\tilde{y}](t)_a\Delta_t^\alpha\eta(t) = \sum_{t=a}^{b-2} ({}_t\Delta_{\rho(b)}^\alpha L_v[\tilde{y}](t)) \eta^\sigma(t). \quad (3.19)$$

Using (3.8) it follows that

$$\begin{aligned} & \sum_{t=a}^{b-1} L_w[\tilde{y}](t)_t\Delta_b^\beta\eta(t) \\ &= - \sum_{t=a}^{b-1} L_w[\tilde{y}](t)\Delta({}_t\Delta_b^{-\nu}\eta(t)) \\ &= - \sum_{t=a}^{b-1} L_w[\tilde{y}](t) \left[{}_t\Delta_{\rho(b)}^{-\nu}\Delta\eta(t) - \frac{\nu}{\Gamma(\nu+1)}(b+\nu-\sigma(t))^{\nu-1}\eta(b) \right] \\ &= - \left(\sum_{t=a}^{b-1} L_w[\tilde{y}](t)_t\Delta_{\rho(b)}^{-\nu}\Delta\eta(t) - \frac{\nu\eta(b)}{\Gamma(\nu+1)} \sum_{t=a}^{b-1} (b+\nu-\sigma(t))^{\nu-1} L_w[\tilde{y}](t) \right). \end{aligned} \quad (3.20)$$

We now use Lemma 54 to get

$$\begin{aligned}
 & \sum_{t=a}^{b-1} L_w[\tilde{y}](t) {}_t\Delta_{\rho(b)}^{-\nu} \Delta\eta(t) \\
 &= \sum_{t=a}^{b-1} L_w[\tilde{y}](t) \Delta\eta(t) + \frac{\nu}{\Gamma(\nu+1)} \sum_{t=a}^{b-2} L_w[\tilde{y}](t) \sum_{s=\sigma(t)}^{b-1} (s+\nu-\sigma(t))^{(\nu-1)} \Delta\eta(s) \\
 &= \sum_{t=a}^{b-1} L_w[\tilde{y}](t) \Delta\eta(t) + \frac{\nu}{\Gamma(\nu+1)} \sum_{t=a}^{b-1} \Delta\eta(t) \sum_{s=a}^{t-1} (t+\nu-\sigma(s))^{(\nu-1)} L_w[\tilde{y}](s) \\
 &= \sum_{t=a}^{b-1} \Delta\eta(t) {}_a\Delta_t^{-\nu} L_w[\tilde{y}](t).
 \end{aligned} \tag{3.21}$$

We apply again the usual summation by parts formula, this time to (3.21), to obtain:

$$\begin{aligned}
 & \sum_{t=a}^{b-1} \Delta\eta(t) {}_a\Delta_t^{-\nu} L_w[\tilde{y}](t) \\
 &= \sum_{t=a}^{b-2} \Delta\eta(t) {}_a\Delta_t^{-\nu} L_w[\tilde{y}](t) + (\eta(b) - \eta(\rho(b))) {}_a\Delta_t^{-\nu} L_w[\tilde{y}](t)|_{t=\rho(b)} \\
 &= [\eta(t) {}_a\Delta_t^{-\nu} L_w[\tilde{y}](t)]_{t=a}^{t=b-1} - \sum_{t=a}^{b-2} \eta^\sigma(t) \Delta({}_a\Delta_t^{-\nu} L_w[\tilde{y}](t)) \\
 &\quad + \eta(b) {}_a\Delta_t^{-\nu} L_w[\tilde{y}](t)|_{t=\rho(b)} - \eta(b-1) {}_a\Delta_t^{-\nu} L_w[\tilde{y}](t)|_{t=\rho(b)} \\
 &= \eta(b) {}_a\Delta_t^{-\nu} L_w[\tilde{y}](t)|_{t=\rho(b)} - \eta(a) {}_a\Delta_t^{-\nu} L_w[\tilde{y}](t)|_{t=a} - \sum_{t=a}^{b-2} \eta^\sigma(t) {}_a\Delta_t^\beta L_w[\tilde{y}](t).
 \end{aligned} \tag{3.22}$$

Since $\eta(a) = \eta(b) = 0$ it follows, from (3.21) and (3.22), that

$$\sum_{t=a}^{b-1} L_w[\tilde{y}](t) {}_t\Delta_{\rho(b)}^{-\nu} \Delta\eta(t) = - \sum_{t=a}^{b-2} \eta^\sigma(t) {}_a\Delta_t^\beta L_w[\tilde{y}](t)$$

and, after inserting in (3.20), that

$$\sum_{t=a}^{b-1} L_w[\tilde{y}](t) {}_t\Delta_b^\beta \eta(t) = \sum_{t=a}^{b-2} \eta^\sigma(t) {}_a\Delta_t^\beta L_w[\tilde{y}](t). \tag{3.23}$$

By (3.19) and (3.23) we may write (3.18) as

$$\sum_{t=a}^{b-2} \left[L_u[\tilde{y}](t) + {}_t\Delta_{\rho(b)}^\alpha L_v[\tilde{y}](t) + {}_a\Delta_t^\beta L_w[\tilde{y}](t) \right] \eta^\sigma(t) = 0.$$

Since the values of $\eta^\sigma(t)$ are arbitrary for $t \in \mathbb{T}^{\kappa^2}$, the Euler–Lagrange equation (3.16) holds along \tilde{y} . \square

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Remark 60. If the initial condition $y(a) = A$ is not present (i.e., $y(a)$ is free), we can use standard techniques to show that the following supplementary condition must be fulfilled:

$$-L_v(a) + \frac{\mu}{\Gamma(\mu+1)} \left(\sum_{t=a}^{b-1} (t+\mu-a)^{(\mu-1)} L_v[\tilde{y}](t) - \sum_{t=\sigma(a)}^{b-1} (t+\mu-\sigma(a))^{(\mu-1)} L_v[\tilde{y}](t) \right) + L_w(a) = 0. \quad (3.24)$$

Similarly, if $y(b) = B$ is not present (i.e., $y(b)$ is free), the equality

$$L_u(\rho(b)) + L_v(\rho(b)) - L_w(\rho(b)) + \frac{\nu}{\Gamma(\nu+1)} \left(\sum_{t=a}^{b-1} (b+\nu-\sigma(t))^{(\nu-1)} L_w[\tilde{y}](t) - \sum_{t=a}^{b-2} (\rho(b)+\nu-\sigma(t))^{(\nu-1)} L_w[\tilde{y}](t) \right) = 0 \quad (3.25)$$

holds. We just note that the first term in (3.25) arises from the first term on the left hand side of (3.18). Equalities (3.24) and (3.25) are the fractional discrete-time *natural boundary conditions*.

The next result is a particular case of our Theorem 59.

Corollary 61 (The discrete-time Euler–Lagrange equation – cf., e.g., [37, 52]). *If \tilde{y} is a solution to the problem*

$$\begin{aligned} \mathcal{L}(y(\cdot)) &= \sum_{t=a}^{b-1} L(t, y(t+1), \Delta y(t)) \longrightarrow \min \\ y(a) &= A, \quad y(b) = B, \end{aligned} \quad (3.26)$$

then $L_u(t, \tilde{y}(t+1), \Delta \tilde{y}(t)) - \Delta L_v(t, \tilde{y}(t+1), \Delta \tilde{y}(t)) = 0$ for all $t \in \{a, \dots, b-2\}$.

Proof. Follows from Theorem 59 with $\alpha = 1$ and a L not depending on w . □

We derive now the second order necessary condition for problem (P), i.e., we obtain Legendre’s necessary condition for the fractional difference setting.

Theorem 62 (The fractional discrete-time Legendre condition). *If $\tilde{y} \in \mathcal{F}$ is a local minimizer for problem (P), then the inequality*

$$\begin{aligned} & L_{uu}[\tilde{y}](t) + 2L_{uv}[\tilde{y}](t) + L_{vv}[\tilde{y}](t) + L_{vv}[\tilde{y}](\sigma(t))(\mu - 1)^2 \\ & + \sum_{s=\sigma(\sigma(t))}^{b-1} L_{vv}[\tilde{y}](s) \left(\frac{\mu(\mu - 1) \prod_{i=0}^{s-t-3} (\mu + i + 1)}{(s - t)\Gamma(s - t)} \right)^2 + 2L_{uv}[\tilde{y}](t)(\nu - 1) \\ & + 2(\nu - 1)L_{vv}[\tilde{y}](t) + 2(\mu - 1)L_{vv}[\tilde{y}](\sigma(t)) + L_{ww}[\tilde{y}](t)(1 - \nu)^2 \\ & + L_{ww}[\tilde{y}](\sigma(t)) + \sum_{s=a}^{t-1} L_{ww}[\tilde{y}](s) \left(\frac{\nu(1 - \nu) \prod_{i=0}^{t-s-2} (\nu + i)}{(\sigma(t) - s)\Gamma(\sigma(t) - s)} \right)^2 \geq 0 \end{aligned}$$

holds for all $t \in \mathbb{T}^{\kappa^2}$, where $[\tilde{y}](t) = (t, \tilde{y}^\sigma(t), {}_a\Delta_t^\alpha \tilde{y}(t), {}_t\Delta_b^\beta \tilde{y}(t))$.

Proof. By the hypothesis of the theorem, and letting Φ be as in (3.17), we get

$$\Phi''(0) \geq 0 \tag{3.27}$$

for an arbitrary admissible variation $\eta(\cdot)$. Inequality (3.27) is equivalent to

$$\begin{aligned} & \sum_{t=a}^{b-1} \left[L_{uu}[\tilde{y}](t)(\eta^\sigma(t))^2 + 2L_{uv}[\tilde{y}](t)\eta^\sigma(t) {}_a\Delta_t^\alpha \eta(t) + L_{vv}[\tilde{y}](t)({}_a\Delta_t^\alpha \eta(t))^2 \right. \\ & \left. + 2L_{uv}[\tilde{y}](t)\eta^\sigma(t) {}_t\Delta_b^\beta \eta(t) + 2L_{vv}[\tilde{y}](t) {}_a\Delta_t^\alpha \eta(t) {}_t\Delta_b^\beta \eta(t) + L_{ww}[\tilde{y}](t)({}_t\Delta_b^\beta \eta(t))^2 \right] \geq 0. \end{aligned}$$

Let $\tau \in \mathbb{T}^{\kappa^2}$ be arbitrary and define $\eta : \mathbb{T} \rightarrow \mathbb{R}$ by

$$\eta(t) = \begin{cases} 1 & \text{if } t = \sigma(\tau); \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\eta(a) = \eta(b) = 0$, i.e., η is an admissible variation. Using (3.7) (note that

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$\eta(a) = 0$), we get

$$\begin{aligned}
& \sum_{t=a}^{b-1} [L_{uu}[\tilde{y}](t)(\eta^\sigma(t))^2 + 2L_{uv}[\tilde{y}](t)\eta^\sigma(t)_a\Delta_t^\alpha\eta(t) + L_{vv}[\tilde{y}](t)({}_a\Delta_t^\alpha\eta(t))^2] \\
&= \sum_{t=a}^{b-1} \left\{ L_{uu}[\tilde{y}](t)(\eta^\sigma(t))^2 \right. \\
&\quad \left. + 2L_{uv}[\tilde{y}](t)\eta^\sigma(t) \left[\Delta\eta(t) + \frac{\mu}{\Gamma(\mu+1)} \sum_{s=a}^{t-1} (t+\mu-\sigma(s))^{\mu-1} \Delta\eta(s) \right] \right. \\
&\quad \left. + L_{vv}[\tilde{y}](t) \left(\Delta\eta(t) + \frac{\mu}{\Gamma(\mu+1)} \sum_{s=a}^{t-1} (t+\mu-\sigma(s))^{\mu-1} \Delta\eta(s) \right)^2 \right\} \\
&= L_{uu}[\tilde{y}](\tau) + 2L_{uv}[\tilde{y}](\tau) + L_{vv}[\tilde{y}](\tau) \\
&\quad + \sum_{t=\sigma(\tau)}^{b-1} L_{vv}[\tilde{y}](t) \left(\Delta\eta(t) + \frac{\mu}{\Gamma(\mu+1)} \sum_{s=a}^{t-1} (t+\mu-\sigma(s))^{\mu-1} \Delta\eta(s) \right)^2.
\end{aligned}$$

Observe that

$$\begin{aligned}
& \sum_{t=\sigma(\sigma(\tau))}^{b-1} L_{vv}[\tilde{y}](t) \left(\frac{\mu}{\Gamma(\mu+1)} \sum_{s=a}^{t-1} (t+\mu-\sigma(s))^{\mu-1} \Delta\eta(s) \right)^2 + L_{vv}(\sigma(\tau))(-1+\mu)^2 \\
&= \sum_{t=\sigma(\tau)}^{b-1} L_{vv}[\tilde{y}](t) \left(\Delta\eta(t) + \frac{\mu}{\Gamma(\mu+1)} \sum_{s=a}^{t-1} (t+\mu-\sigma(s))^{\mu-1} \Delta\eta(s) \right)^2.
\end{aligned}$$

We show next that

$$\begin{aligned}
& \sum_{t=\sigma(\sigma(\tau))}^{b-1} L_{vv}[\tilde{y}](t) \left(\frac{\mu}{\Gamma(\mu+1)} \sum_{s=a}^{t-1} (t+\mu-\sigma(s))^{\mu-1} \Delta\eta(s) \right)^2 \\
&= \sum_{t=\sigma(\sigma(\tau))}^{b-1} L_{vv}[\tilde{y}](t) \left(\frac{\mu(\mu-1) \prod_{i=0}^{t-\tau-3} (\mu+i+1)}{(t-\tau)\Gamma(t-\tau)} \right)^2.
\end{aligned}$$

Let $t \in [\sigma(\sigma(\tau)), b-1] \cap \mathbb{Z}$. Then,

$$\begin{aligned}
 & \frac{\mu}{\Gamma(\mu+1)} \sum_{s=a}^{t-1} (t+\mu-\sigma(s))^{\mu-1} \Delta\eta(s) \\
 &= \frac{\mu}{\Gamma(\mu+1)} \left[\sum_{s=a}^{\tau} (t+\mu-\sigma(s))^{\mu-1} \Delta\eta(s) + \sum_{s=\sigma(\tau)}^{t-1} (t+\mu-\sigma(s))^{\mu-1} \Delta\eta(s) \right] \\
 &= \frac{\mu}{\Gamma(\mu+1)} \left[(t+\mu-\sigma(\tau))^{\mu-1} - (t+\mu-\sigma(\sigma(\tau)))^{\mu-1} \right] \\
 &= \frac{\mu}{\Gamma(\mu+1)} \left[\frac{\Gamma(t+\mu-\sigma(\tau)+1)}{\Gamma(t+\mu-\sigma(\tau)+1-(\mu-1))} - \frac{\Gamma(t+\mu-\sigma(\sigma(\tau))+1)}{\Gamma(t+\mu-\sigma(\sigma(\tau))+1-(\mu-1))} \right] \\
 &= \frac{\mu}{\Gamma(\mu+1)} \left[\frac{\Gamma(t+\mu-\tau)}{\Gamma(t-\tau+1)} - \frac{\Gamma(t-\tau+\mu-1)}{\Gamma(t-\tau)} \right] \\
 &= \frac{\mu}{\Gamma(\mu+1)} \left[\frac{(t+\mu-\tau-1)\Gamma(t+\mu-\tau-1)}{(t-\tau)\Gamma(t-\tau)} - \frac{(t-\tau)\Gamma(t-\tau+\mu-1)}{(t-\tau)\Gamma(t-\tau)} \right] \\
 &= \frac{\mu}{\Gamma(\mu+1)} \frac{(\mu-1)\Gamma(t-\tau+\mu-1)}{(t-\tau)\Gamma(t-\tau)} \\
 &= \frac{\mu(\mu-1) \prod_{i=0}^{t-\tau-3} (\mu+i+1)}{(t-\tau)\Gamma(t-\tau)}, \tag{3.28}
 \end{aligned}$$

which proves our claim. Observe that we can write ${}_t\Delta_b^\beta\eta(t) = -{}_t\Delta_{\rho(b)}^{-\nu}\Delta\eta(t)$ since $\eta(b) = 0$. It is not difficult to see that the following equality holds:

$$\sum_{t=a}^{b-1} 2L_{uw}[\tilde{y}](t)\eta^\sigma(t) {}_t\Delta_b^\beta\eta(t) = - \sum_{t=a}^{b-1} 2L_{uw}[\tilde{y}](t)\eta^\sigma(t) {}_t\Delta_{\rho(b)}^{-\nu}\Delta\eta(t) = 2L_{uw}[\tilde{y}](\tau)(\nu-1).$$

Moreover,

$$\begin{aligned}
 & \sum_{t=a}^{b-1} 2L_{vw}[\tilde{y}](t) {}_a\Delta_t^\alpha\eta(t) {}_t\Delta_b^\beta\eta(t) \\
 &= -2 \sum_{t=a}^{b-1} L_{vw}[\tilde{y}](t) \left\{ \left(\Delta\eta(t) + \frac{\mu}{\Gamma(\mu+1)} \cdot \sum_{s=a}^{t-1} (t+\mu-\sigma(s))^{\mu-1} \Delta\eta(s) \right) \right. \\
 & \quad \left. \cdot \left[\Delta\eta(t) + \frac{\nu}{\Gamma(\nu+1)} \sum_{s=\sigma(t)}^{b-1} (s+\nu-\sigma(t))^{\nu-1} \Delta\eta(s) \right] \right\} \\
 &= 2(\nu-1)L_{vw}[\tilde{y}](\tau) + 2(\mu-1)L_{vw}[\tilde{y}](\sigma(\tau)).
 \end{aligned}$$

Finally, we have that

$$\begin{aligned}
& \sum_{t=a}^{b-1} L_{ww}[\tilde{y}](t) ({}_t\Delta_b^\beta \eta(t))^2 \\
&= \sum_{t=a}^{\sigma(\tau)} L_{ww}[\tilde{y}](t) \left[\Delta\eta(t) + \frac{\nu}{\Gamma(\nu+1)} \sum_{s=\sigma(t)}^{b-1} (s+\nu-\sigma(t))^{\nu-1} \Delta\eta(s) \right]^2 \\
&= \sum_{t=a}^{\tau-1} L_{ww}[\tilde{y}](t) \left[\frac{\nu}{\Gamma(\nu+1)} \sum_{s=\sigma(t)}^{b-1} (s+\nu-\sigma(t))^{\nu-1} \Delta\eta(s) \right]^2 \\
&\quad + L_{ww}[\tilde{y}](\tau)(1-\nu)^2 + L_{ww}[\tilde{y}](\sigma(\tau)) \\
&= \sum_{t=a}^{\tau-1} L_{ww}[\tilde{y}](t) \left[\frac{\nu}{\Gamma(\nu+1)} \left\{ (\tau+\nu-\sigma(t))^{\nu-1} - (\sigma(\tau)+\nu-\sigma(t))^{\nu-1} \right\} \right]^2 \\
&\quad + L_{ww}[\tilde{y}](\tau)(1-\nu)^2 + L_{ww}[\tilde{y}](\sigma(\tau)).
\end{aligned}$$

Similarly as we have done in (3.28), we obtain that

$$\frac{\nu}{\Gamma(\nu+1)} \left[(\tau+\nu-\sigma(t))^{\nu-1} - (\sigma(\tau)+\nu-\sigma(t))^{\nu-1} \right] = \frac{\nu(1-\nu) \prod_{i=0}^{\tau-t-2} (\nu+i)}{(\sigma(\tau)-t)\Gamma(\sigma(\tau)-t)}.$$

We are done with the proof. \square

A trivial corollary of our result gives the discrete-time version of Legendre's necessary condition.

Corollary 63 (The discrete-time Legendre condition – cf., e.g., [37,64]). *If \tilde{y} is a solution to the problem (3.26), then*

$$L_{uu}[\tilde{y}](t) + 2L_{uv}[\tilde{y}](t) + L_{vv}[\tilde{y}](t) + L_{vv}[\tilde{y}](\sigma(t)) \geq 0$$

holds for all $t \in \mathbb{T}^{\kappa^2}$, where $[\tilde{y}](t) = (t, \tilde{y}^\sigma(t), \Delta\tilde{y}(t))$.

Proof. We consider problem (P) with $\alpha = 1$ and L not depending on w . The choice $\alpha = 1$ implies $\mu = 0$, and the result follows immediately from Theorem 62. \square

3.4 Examples

In this section we present three illustrative examples. The results were obtained using the open source Computer Algebra System Maxima.¹ All computations were done running

¹<http://maxima.sourceforge.net>

Maxima on an Intel® Core™2 Duo, CPU of 2.27GHz with 3Gb of RAM. Our Maxima definitions are given in Appendix A.

Example 64. Let us consider the following problem:

$$J_\alpha(y) = \sum_{t=0}^{b-1} ({}_0\Delta_t^\alpha y(t))^2 \longrightarrow \min, \quad y(0) = A, \quad y(b) = B. \quad (3.29)$$

In this case Theorem 62 is trivially satisfied. We obtain the solution \tilde{y} to our Euler–Lagrange equation (3.16) for the case $b = 2$ using the computer algebra system Maxima. Using our Maxima package (see the definition of the command `extremal` in Appendix A) we do

```
L1:v^2$
extremal(L1,0,2,A,B,alpha,alpha);
```

to obtain (2 seconds)

$$\tilde{y}(1) = \frac{2\alpha B + (\alpha^3 - \alpha^2 + 2\alpha) A}{2\alpha^2 + 2}. \quad (3.30)$$

For the particular case $\alpha = 1$ the equality (3.30) gives $\tilde{y}(1) = \frac{A+B}{2}$, which coincides with the solution to the (non-fractional) discrete problem

$$\sum_{t=0}^1 (\Delta y(t))^2 = \sum_{t=0}^1 (y(t+1) - y(t))^2 \longrightarrow \min, \quad y(0) = A, \quad y(2) = B.$$

Similarly, we can obtain exact formulas of the extremal on bigger intervals (for bigger values of b). For example, the solution of problem (3.29) with $b = 3$ is (35 seconds)

$$\begin{aligned} \tilde{y}(1) &= \frac{(6\alpha^2 + 6\alpha) B + (2\alpha^5 + 2\alpha^4 + 10\alpha^3 - 2\alpha^2 + 12\alpha) A}{3\alpha^4 + 6\alpha^3 + 15\alpha^2 + 12}, \\ \tilde{y}(2) &= \frac{(12\alpha^3 + 12\alpha^2 + 24\alpha) B + (\alpha^6 + \alpha^5 + 7\alpha^4 - \alpha^3 + 4\alpha^2 + 12\alpha) A}{6\alpha^4 + 12\alpha^3 + 30\alpha^2 + 24}; \end{aligned}$$

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and the solution of problem (3.29) with $b = 4$ is (72 seconds)

$$\begin{aligned}\tilde{y}(1) &= \frac{3\alpha^7 + 15\alpha^6 + 57\alpha^5 + 69\alpha^4 + 156\alpha^3 - 12\alpha^2 + 144\alpha}{\xi}A \\ &\quad + \frac{24\alpha^3 + 72\alpha^2 + 48\alpha}{\xi}B, \\ \tilde{y}(2) &= \frac{\alpha^8 + 5\alpha^7 + 22\alpha^6 + 32\alpha^5 + 67\alpha^4 + 35\alpha^3 + 54\alpha^2 + 72\alpha}{\xi}A \\ &\quad + \frac{24\alpha^4 + 72\alpha^3 + 120\alpha^2 + 72\alpha}{\xi}B, \\ \tilde{y}(3) &= \frac{\alpha^9 + 6\alpha^8 + 30\alpha^7 + 60\alpha^6 + 117\alpha^5 + 150\alpha^4 - 4\alpha^3 + 216\alpha^2 + 288\alpha}{\zeta}A \\ &\quad + \frac{72\alpha^5 + 288\alpha^4 + 792\alpha^3 + 576\alpha^2 + 864\alpha}{\zeta}B,\end{aligned}$$

where

$$\begin{aligned}\xi &= 4\alpha^6 + 24\alpha^5 + 88\alpha^4 + 120\alpha^3 + 196\alpha^2 + 144, \\ \zeta &= 24\alpha^6 + 144\alpha^5 + 528\alpha^4 + 720\alpha^3 + 1176\alpha^2 + 864.\end{aligned}$$

Consider now problem (3.29) with $b = 4$, $A = 0$, and $B = 1$. In Table 3.1 we show the extremal values $\tilde{y}(1)$, $\tilde{y}(2)$, $\tilde{y}(3)$, and corresponding \tilde{J}_α , for some values of α . Our numerical results show that the fractional extremal converges to the classical (integer order) extremal when α tends to one. This is illustrated in Figure 3.1. The numerical results from Table 3.1 and Figure 3.2 show that for this problem the smallest value of \tilde{J}_α , $\alpha \in]0, 1]$, occur for $\alpha = 1$ (i.e., the smallest value of \tilde{J}_α occurs for the classical non-fractional case).

α	$\tilde{y}(1)$	$\tilde{y}(2)$	$\tilde{y}(3)$	\tilde{J}_α
0.25	0.10647146897355	0.16857982587479	0.2792657904952	0.90855653524095
0.50	0.20997375328084	0.35695538057743	0.54068241469816	0.67191601049869
0.75	0.25543605027861	0.4702345471038	0.69508876506414	0.4246209666969
1	0.25	0.5	0.75	0.25

Table 3.1: The extremal values $\tilde{y}(1)$, $\tilde{y}(2)$ and $\tilde{y}(3)$ of problem (3.29) with $b = 4$, $A = 0$, and $B = 1$ for different α 's.

Example 65. In this example we generalize problem (3.29) to

$$J_{\alpha,\beta} = \sum_{t=0}^{b-1} \gamma_1 \left({}_0\Delta_t^\alpha y(t) \right)^2 + \gamma_2 \left({}_t\Delta_b^\beta y(t) \right)^2 \longrightarrow \min, \quad y(0) = A, \quad y(b) = B. \quad (3.31)$$

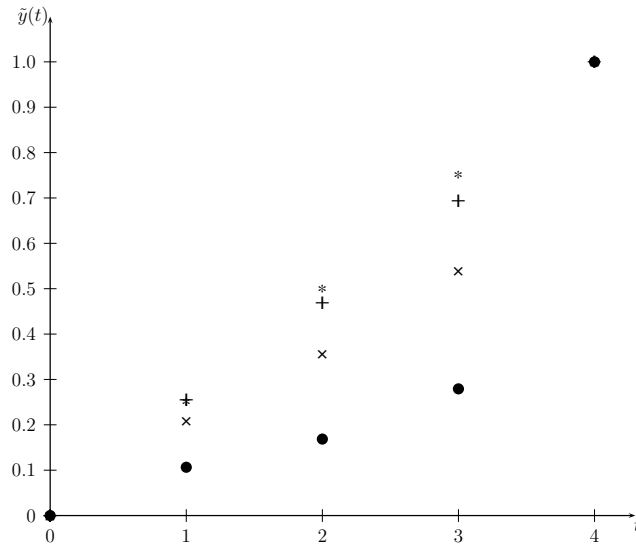


Figure 3.1: Extremal $\tilde{y}(t)$ of Example 64 with $b = 4$, $A = 0$, $B = 1$, and different α 's (●: $\alpha = 0.25$; ×: $\alpha = 0.5$; +: $\alpha = 0.75$; *: $\alpha = 1$).

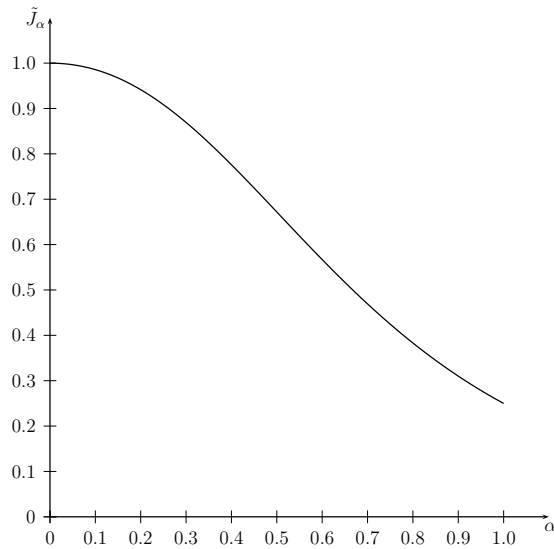


Figure 3.2: Function \tilde{J}_α of Example 64 with $b = 4$, $A = 0$, and $B = 1$.

3.4. EXAMPLES

As before, we solve the associated Euler–Lagrange equation (3.16) for the case $b = 2$ with the help of our Maxima package (35 seconds):

```
L2: (gamma[1])*v^2+(gamma[2])*w^2$
extremal(L2,0,2,A,B,alpha,beta);
```

$$\tilde{y}(1) = \frac{(2\gamma_2\beta + \gamma_1\alpha^3 - \gamma_1\alpha^2 + 2\gamma_1\alpha)A + (\gamma_2\beta^3 - \gamma_2\beta^2 + 2\gamma_2\beta + 2\gamma_1\alpha)B}{2\gamma_2\beta^2 + 2\gamma_1\alpha^2 + 2\gamma_2 + 2\gamma_1}.$$

Consider now problem (3.31) with $\gamma_1 = \gamma_2 = 1$, $b = 2$, $A = 0$, $B = 1$, and $\beta = \alpha$. In Table 3.2 we show the values of $\tilde{y}(1)$ and $\tilde{J}_\alpha := J_{\alpha,\alpha}(\tilde{y}(1))$ for some values of α . We concluded, numerically, that the fractional extremal $\tilde{y}(1)$ tends to the classical (non-fractional) extremal when α tends to one. Differently from Example 64, the smallest value of \tilde{J}_α , $\alpha \in]0, 1]$, does not occur here for $\alpha = 1$ (see Figure 3.3). The smallest value of \tilde{J}_α , $\alpha \in]0, 1]$, occurs for $\alpha = 0.61747447161482$.

α	$\tilde{y}(1)$	\tilde{J}_α
0.25	0.22426470588235	0.96441291360294
0.50	0.375	0.9140625
0.75	0.4575	0.91720703125
1	0.5	1

Table 3.2: The extremal $\tilde{y}(1)$ of problem (3.31) for different values of α ($\gamma_1 = \gamma_2 = 1$, $b = 2$, $A = 0$, $B = 1$, and $\beta = \alpha$).

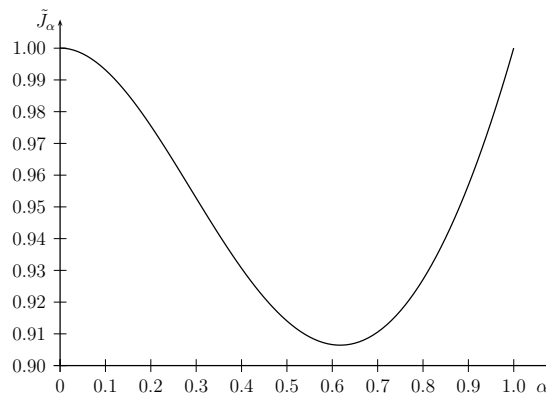


Figure 3.3: Function \tilde{J}_α of Example 65 with $\gamma_1 = \gamma_2 = 1$, $b = 2$, $A = 0$, $B = 1$, and $\beta = \alpha$.

Example 66. Our last example is a discrete version of the fractional continuous problem [6, Example 2]:

$$J_\alpha = \sum_{t=0}^1 \frac{1}{2} ({}_0\Delta_t^\alpha y(t))^2 - y^\sigma(t) \longrightarrow \min, \quad y(0) = 0, \quad y(2) = 0. \quad (3.32)$$

The Euler–Lagrange extremal of (3.32) is easily obtained with our Maxima package (4 seconds):

```
L3: (1/2)*v^2-u;$
extremal(L3,0,2,0,0,alpha,beta);
```

$$\tilde{y}(1) = \frac{1}{\alpha^2 + 1}. \quad (3.33)$$

For the particular case $\alpha = 1$ the equality (3.33) gives $\tilde{y}(1) = \frac{1}{2}$, which coincides with the solution to the non-fractional discrete problem

$$\sum_{t=0}^1 \frac{1}{2} (\Delta y(t))^2 - y^\sigma(t) = \sum_{t=0}^1 \frac{1}{2} (y(t+1) - y(t))^2 - y(t+1) \longrightarrow \min,$$

$$y(0) = 0, \quad y(2) = 0.$$

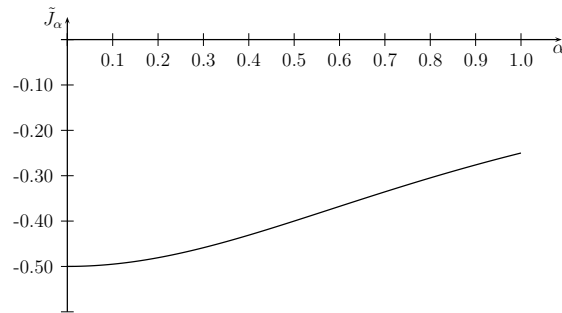
In Table 3.3 we show the values of $\tilde{y}(1)$ and \tilde{J}_α for some α 's. As seen in Figure 3.4, for $\alpha = 1$ one gets the maximum value of \tilde{J}_α , $\alpha \in]0, 1]$.

α	$\tilde{y}(1)$	\tilde{J}_α
0.25	0.94117647058824	-0.47058823529412
0.50	0.8	-0.4
0.75	0.64	-0.32
1	0.5	-0.25

Table 3.3: Extremal values $\tilde{y}(1)$ of (3.32) for different α 's

3.5 Conclusion

In this chapter we introduce the study of fractional discrete-time problems of the calculus of variations of order α , $0 < \alpha \leq 1$, with left and right discrete operators of Riemann–Liouville type. For $\alpha = 1$ we obtain the classical discrete-time results of the calculus of variations [71].

Figure 3.4: Function \tilde{J}_α of Example 66.

Main results of the chapter include a fractional summation by parts formula (Theorem 55), a fractional discrete-time Euler–Lagrange equation (Theorem 59), transversality conditions (3.24) and (3.25), and a fractional discrete-time Legendre condition (Theorem 62). From the analysis of the results obtained from computer experiments, we conclude that when the value of α approaches one, the optimal value of the fractional discrete functional converges to the optimal value of the classical (non-fractional) discrete problem. On the other hand, the value of α for which the functional attains its minimum varies with the concrete problem under consideration.

3.6 State of the Art

The discrete-time calculus is also a very important tool in practical applications and in the modeling of real phenomena. Therefore, it is not a surprise that fractional discrete calculus is recently under strong development (see, e.g., [28, 34, 46, 58, 59] and references therein). There are some recent theses [67, 109] with results related with discrete calculus on time scale $\mathbb{T} = \{a, a + 1, a + 2, \dots\}$. Thesis [109] links the use of fractional difference equations with tumor growth modeling, and addresses a very important question: by changing the order of the difference equations from integers to fractional, in which conditions, we are able to provide more accurate models for real world problems? For more on the subject see [101].

In this chapter we use fractional difference operators of Riemann–Liouville type. However, there are some results related with fractional difference operators in Caputo sense [17] like

in [45], where the existence of solutions for IVP involving nonlinear fractional difference equations is discussed.

The results of this chapter are published in [33] and were presented by the author at the Workshop on Control, Nonsmooth Analysis and Optimization, Celebrating Francis Clarke's and Richard Vinter's 60th Birthday, Porto, Portugal, May 4–8, 2009 in a contributed talk entitled *Necessary Optimality Conditions for Fractional Difference Problems of the Calculus of Variations*.

Chapter 4

Fractional Variational Problems in

$$\mathbb{T} = (h\mathbb{Z})_a$$

We introduce a discrete-time fractional calculus of variations on the time scale $(h\mathbb{Z})_a$, $h > 0$. First and second order necessary optimality conditions are established. Examples illustrating the use of the new Euler-Lagrange and Legendre type conditions are given.

4.1 Introduction

Although the fractional Euler–Lagrange equations are obtained in a similar manner as in the standard variational calculus [102], some classical results are extremely difficult to be proved in a fractional context. This explains, for example, why a fractional Legendre type condition is absent from the literature of fractional variational calculus. In this chapter we give a first result in this direction (cf. Theorem 91).

Despite its importance in applications, less is known for discrete-time fractional systems [102]. Our objective is proceed to develop the theory of *fractional difference calculus*, namely, we introduce the concept of left and right fractional sum/difference (cf. Definition 76). In Section 4.2 we introduce notations, we give necessary definitions, and prove some preliminary results needed in the sequel. Main results of the paper appear in Section 4.3: we prove a fractional formula of h -summation by parts (Theorem 85), and necessary optimality conditions of first and second order (Theorems 88 and 91, respectively) for the proposed h -fractional problem of the calculus of variations (4.15). Before the end of the chapter we present in Section 4.4 some illustrative examples and give some conclusions

in Section 4.5 and the state of the art in Section 4.6.

4.2 Preliminaries

One way to approach the Riemann-Liouville fractional calculus is through the theory of linear differential equations [100]. Miller and Ross [86] use an analogous methodology to introduce fractional discrete operators for the case $\mathbb{T} = \mathbb{Z} = \{a, a + 1, a + 2, \dots\}$, $a \in \mathbb{R}$. That was our starting point for the results presented in Chapter 3. Because the results obtained in the previous chapter were good, we start thinking how to generalize them to an arbitrary time scale. However, since we found several difficulties when we try to get results for a time-scale where the graininess function is not constant, here we keep the graininess function constant but not, necessarily, equal to one and we give a step further: we use the theory of time scales in order to introduce fractional discrete operators to the more general case $\mathbb{T} = (h\mathbb{Z})_a = \{a, a + h, a + 2h, \dots\}$, $a \in \mathbb{R}$, $h > 0$.

Before going further we need to collect more results on the theory of time-scales.

Until we state the opposite, the following results are true for a arbitrary time scale.

For $n \in \mathbb{N}_0$ and rd-continuous functions $p_i : \mathbb{T} \rightarrow \mathbb{R}$, $1 \leq i \leq n$, let us consider the n th order linear dynamic equation

$$Ly = 0, \quad \text{where } Ly = y^{\Delta^n} + \sum_{i=1}^n p_i y^{\Delta^{n-i}}. \quad (4.1)$$

A function $y : \mathbb{T} \rightarrow \mathbb{R}$ is said to be a solution of equation (4.1) on \mathbb{T} provided y is n times delta differentiable on \mathbb{T}^{κ^n} and satisfies $Ly(t) = 0$ for all $t \in \mathbb{T}^{\kappa^n}$.

Lemma 67. [41, p. 239] *If $z = (z_1, \dots, z_n) : \mathbb{T} \rightarrow \mathbb{R}^n$ satisfies for all $t \in \mathbb{T}^{\kappa}$*

$$z^{\Delta} = A(t)z(t), \quad \text{where } A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \\ -p_n & \dots & \dots & -p_2 & -p_1 \end{pmatrix} \quad (4.2)$$

then $y = z_1$ is a solution of equation (4.1). Conversely, if y solves (4.1) on \mathbb{T} , then $z = (y, y^{\Delta}, \dots, y^{\Delta^{n-1}}) : \mathbb{T} \rightarrow \mathbb{R}^n$ satisfies (4.2) for all $t \in \mathbb{T}^{\kappa^n}$

Definition 68. [41, p. 239] We say that equation (4.1) is *regressive* provided $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}^\kappa$, where A is the matrix in (4.2).

Definition 69. [41, p. 250] We define the Cauchy function $y : \mathbb{T} \times \mathbb{T}^{\kappa^n} \rightarrow \mathbb{R}$ for the linear dynamic equation (4.1) to be, for each fixed $s \in \mathbb{T}^{\kappa^n}$, the solution of the initial value problem

$$Ly = 0, \quad y^{\Delta^i}(\sigma(s), s) = 0, \quad 0 \leq i \leq n-2, \quad y^{\Delta^{n-1}}(\sigma(s), s) = 1. \quad (4.3)$$

Theorem 70. [41, p. 251] Suppose $\{y_1, \dots, y_n\}$ is a fundamental system of the regressive equation (4.1). Let $f \in C_{rd}$. Then the solution of the initial value problem

$$Ly = f(t), \quad y^{\Delta^i}(t_0) = 0, \quad 0 \leq i \leq n-1,$$

is given by $y(t) = \int_{t_0}^t y(t, s)f(s)\Delta s$, where $y(t, s)$ is the Cauchy function for (4.1).

It is known that $y(t, s) := H_{n-1}(t, \sigma(s))$ is the Cauchy function for $y^{\Delta^n} = 0$, where H_{n-1} is a time scale generalized polynomial [41, Example 5.115]. The generalized polynomials H_k are the functions $H_k : \mathbb{T}^2 \rightarrow \mathbb{R}$, $k \in \mathbb{N}_0$, defined recursively as follows:

$$H_0(t, s) \equiv 1, \quad H_{k+1}(t, s) = \int_s^t H_k(\tau, s)\Delta\tau, \quad k = 1, 2, \dots$$

for all $s, t \in \mathbb{T}$. If we let $H_k^\Delta(t, s)$ denote, for each fixed s , the derivative of $H_k(t, s)$ with respect to t , then (cf. [41, p. 38])

$$H_k^\Delta(t, s) = H_{k-1}(t, s) \quad \text{for } k \in \mathbb{N}, \quad t \in \mathbb{T}^\kappa.$$

Let $a \in \mathbb{R}$ and $h > 0$, $(h\mathbb{Z})_a = \{a, a+h, a+2h, \dots\}$, and $b = a + kh$ for some $k \in \mathbb{N}$.

From now on we restrict ourselves to the time scale $\mathbb{T} = (h\mathbb{Z})_a$, $h > 0$. Our main goal is to propose and develop a discrete-time fractional variational theory in $\mathbb{T} = (h\mathbb{Z})_a$. We borrow the notations from the recent calculus of variations on time scales [37, 52, 69]. How to generalize our results to an arbitrary time scale \mathbb{T} , with the graininess function μ depending on time, is not clear and remains a challenging question.

We have $\sigma(t) = t+h$, $\rho(t) = t-h$, $\mu(t) \equiv h$, and we will frequently write $f^\sigma(t) = f(\sigma(t))$. We put $\mathbb{T} = [a, b] \cap (h\mathbb{Z})_a$, so that $\mathbb{T}^\kappa = [a, \rho(b)] \cap (h\mathbb{Z})_a$ and $\mathbb{T}^{\kappa^2} = [a, \rho^2(b)] \cap (h\mathbb{Z})_a$.

The delta derivative coincides in this case with the forward h -difference:

$$f^\Delta(t) = \frac{f^\sigma(t) - f(t)}{\mu(t)}.$$

If $h = 1$, then we have the usual discrete forward difference $\Delta f(t)$.

The delta integral gives the h -sum (or h -integral) of f : $\int_a^b f(t)\Delta t = \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h$. If we have a function f of two variables, $f(t, s)$, its partial forward h -differences will be denoted by $\Delta_{t,h}$ and $\Delta_{s,h}$, respectively. We make use of the standard conventions $\sum_{t=c}^{c-1} f(t) = 0$, $c \in \mathbb{Z}$, and $\prod_{i=0}^{-1} f(i) = 1$. Often, *left fractional delta integration* (resp., *right fractional delta integration*) of order $\nu > 0$ is denoted by ${}_a\Delta_t^{-\nu} f(t)$ (resp. ${}_t\Delta_b^{-\nu} f(t)$). Here, similarly as in Ross *et al.* [107], where the authors omit the subscript t on the operator (the operator itself cannot depend on t), we write ${}_a\Delta_h^{-\nu} f(t)$ (resp. ${}_h\Delta_b^{-\nu} f(t)$).

Before giving an explicit formula for the generalized polynomials H_k on $h\mathbb{Z}$ we introduce the following definition:

Definition 71. For arbitrary $x, y \in \mathbb{R}$ the h -factorial function is defined by

$$x_h^{(y)} := h^y \frac{\Gamma(\frac{x}{h} + 1)}{\Gamma(\frac{x}{h} + 1 - y)},$$

where Γ is the well-known Euler gamma function, and we use the convention that division at a pole yields zero.

Remark 72. Before proposing Definition 71 we have tried other possibilities. One that seemed most obvious is just to replace 1 by h in formula (3.2) because if we do $h = 1$ on the time scale $(h\mathbb{Z})_a$ we have the time scale \mathbb{Z} . This possibility did not reveal a good choice.

Remark 73. For $h = 1$, and in accordance with the previous similar definition on Chapter 3, we write $x^{(y)}$ to denote $x_h^{(y)}$.

Proposition 74. For the time-scale $\mathbb{T} = (h\mathbb{Z})_a$ one has

$$H_k(t, s) := \frac{(t-s)_h^{(k)}}{k!} \quad \text{for all } s, t \in \mathbb{T} \text{ and } k \in \mathbb{N}_0. \quad (4.4)$$

To prove (4.4) we use the following technical lemma. Throughout this chapter the basic property (1.6) of the gamma function will be frequently used.

Lemma 75. Let $s \in \mathbb{T}$. Then, for all $t \in \mathbb{T}^\kappa$ one has

$$\Delta_{t,h} \left\{ \frac{(t-s)_h^{(k+1)}}{(k+1)!} \right\} = \frac{(t-s)_h^{(k)}}{k!}.$$

Proof. The equality follows by direct computations:

$$\begin{aligned}
 \Delta_{t,h} \left\{ \frac{(t-s)_h^{(k+1)}}{(k+1)!} \right\} &= \frac{1}{h} \left\{ \frac{(\sigma(t)-s)_h^{(k+1)}}{(k+1)!} - \frac{(t-s)_h^{(k+1)}}{(k+1)!} \right\} \\
 &= \frac{h^{k+1}}{h(k+1)!} \left\{ \frac{\Gamma((t+h-s)/h+1)}{\Gamma((t+h-s)/h+1-(k+1))} - \frac{\Gamma((t-s)/h+1)}{\Gamma((t-s)/h+1-(k+1))} \right\} \\
 &= \frac{h^k}{(k+1)!} \left\{ \frac{((t-s)/h+1)\Gamma((t-s)/h+1)}{((t-s)/h-k)\Gamma((t-s)/h-k)} - \frac{\Gamma((t-s)/h+1)}{\Gamma((t-s)/h-k)} \right\} \\
 &= \frac{h^k}{k!} \left\{ \frac{\Gamma((t-s)/h+1)}{\Gamma((t-s)/h+1-k)} \right\} = \frac{(t-s)_h^{(k)}}{k!}.
 \end{aligned}$$

□

Proof. (of Proposition 74) We proceed by mathematical induction. For $k = 0$

$$H_0(t, s) = \frac{1}{0!} h^0 \frac{\Gamma(\frac{t-s}{h} + 1)}{\Gamma(\frac{t-s}{h} + 1 - 0)} = \frac{\Gamma(\frac{t-s}{h} + 1)}{\Gamma(\frac{t-s}{h} + 1)} = 1.$$

Assume that (4.4) holds for k replaced by m . Then by Lemma 75

$$H_{m+1}(t, s) = \int_s^t H_m(\tau, s) \Delta \tau = \int_s^t \frac{(\tau-s)_h^{(m)}}{m!} \Delta \tau = \frac{(t-s)_h^{(m+1)}}{(m+1)!},$$

which is (4.4) with k replaced by $m+1$. □

Let $y_1(t), \dots, y_n(t)$ be n linearly independent solutions of the linear homogeneous dynamic equation $y^{\Delta^n} = 0$. From Theorem 70 we know that the solution of (4.3) (with $L = \Delta^n$ and $t_0 = a$) is

$$y(t) = \Delta^{-n} f(t) = \int_a^t \frac{(t-\sigma(s))_h^{(n-1)}}{\Gamma(n)} f(s) \Delta s = \frac{1}{\Gamma(n)} \sum_{k=a/h}^{t/h-1} (t-\sigma(kh))_h^{(n-1)} f(kh) h.$$

Since $y^{\Delta^i}(a) = 0$, $i = 0, \dots, n-1$, then we can write that

$$\begin{aligned}
 \Delta^{-n} f(t) &= \frac{1}{\Gamma(n)} \sum_{k=a/h}^{t/h-n} (t-\sigma(kh))_h^{(n-1)} f(kh) h \\
 &= \frac{1}{\Gamma(n)} \int_a^{\sigma(t-nh)} (t-\sigma(s))_h^{(n-1)} f(s) \Delta s.
 \end{aligned} \tag{4.5}$$

Note that function $t \rightarrow (\Delta^{-n} f)(t)$ is defined for $t = a + nh \pmod{h}$ while function $t \rightarrow f(t)$ is defined for $t = a \pmod{h}$. Extending (4.5) to any positive real value ν , and having as an

analogy the continuous left and right fractional derivatives [87], we define the left fractional h -sum and the right fractional h -sum as follows. We denote by $\mathcal{F}_{\mathbb{T}}$ the set of all real valued functions defined on a given time scale \mathbb{T} .

Definition 76. Let $a \in \mathbb{R}$, $h > 0$, $b = a + kh$ with $k \in \mathbb{N}$, and put $\mathbb{T} = [a, b] \cap (h\mathbb{Z})_a$. Consider $f \in \mathcal{F}_{\mathbb{T}}$. The left and right fractional h -sum of order $\nu > 0$ are, respectively, the operators ${}_a\Delta_h^{-\nu} : \mathcal{F}_{\mathbb{T}} \rightarrow \mathcal{F}_{\tilde{\mathbb{T}}_h^+}$ and ${}_h\Delta_b^{-\nu} : \mathcal{F}_{\mathbb{T}} \rightarrow \mathcal{F}_{\tilde{\mathbb{T}}_h^-}$, $\tilde{\mathbb{T}}_{\nu}^{\pm} = \{t \pm \nu h : t \in \mathbb{T}\}$, defined by

$${}_a\Delta_h^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_a^{\sigma(t-\nu h)} (t - \sigma(s))_h^{(\nu-1)} f(s) \Delta s = \frac{1}{\Gamma(\nu)} \sum_{k=\frac{a}{h}}^{\frac{t}{h}-\nu} (t - \sigma(kh))_h^{(\nu-1)} f(kh)h$$

$${}_h\Delta_b^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_{t+\nu h}^{\sigma(b)} (s - \sigma(t))_h^{(\nu-1)} f(s) \Delta s = \frac{1}{\Gamma(\nu)} \sum_{k=\frac{t}{h}+\nu}^{\frac{b}{h}} (kh - \sigma(t))_h^{(\nu-1)} f(kh)h.$$

Remark 77. In Definition 76 we are using summations with limits that are reals. For example, the summation that appears in the definition of operator ${}_a\Delta_h^{-\nu}$ has the following meaning:

$$\sum_{k=\frac{a}{h}}^{\frac{t}{h}-\nu} G(k) = G(a/h) + G(a/h + 1) + G(a/h + 2) + \cdots + G(t/h - \nu),$$

where $t \in \{a + \nu h, a + h + \nu h, a + 2h + \nu h, \dots, \underbrace{a + kh + \nu h}_b\}$ with $k \in \mathbb{N}$.

Lemma 78. Let $\nu > 0$ be an arbitrary positive real number. For any $t \in \mathbb{T}$ we have: (i) $\lim_{\nu \rightarrow 0} {}_a\Delta_h^{-\nu} f(t + \nu h) = f(t)$; (ii) $\lim_{\nu \rightarrow 0} {}_h\Delta_b^{-\nu} f(t - \nu h) = f(t)$.

Proof. Since

$$\begin{aligned} {}_a\Delta_h^{-\nu} f(t + \nu h) &= \frac{1}{\Gamma(\nu)} \int_a^{\sigma(t)} (t + \nu h - \sigma(s))_h^{(\nu-1)} f(s) \Delta s \\ &= \frac{1}{\Gamma(\nu)} \sum_{k=\frac{a}{h}}^{\frac{t}{h}} (t + \nu h - \sigma(kh))_h^{(\nu-1)} f(kh)h \\ &= h^{\nu} f(t) + \frac{\nu}{\Gamma(\nu + 1)} \sum_{k=\frac{a}{h}}^{\frac{\rho(t)}{h}} (t + \nu h - \sigma(kh))_h^{(\nu-1)} f(kh)h, \end{aligned}$$

it follows that $\lim_{\nu \rightarrow 0} {}_a\Delta_h^{-\nu} f(t + \nu h) = f(t)$. The proof of (ii) is similar. \square

4.2. PRELIMINARIES

For any $t \in \mathbb{T}$ and for any $\nu \geq 0$ we define ${}_a\Delta_h^0 f(t) := {}_h\Delta_b^0 f(t) := f(t)$ and write

$$\begin{aligned} {}_a\Delta_h^{-\nu} f(t + \nu h) &= h^\nu f(t) + \frac{\nu}{\Gamma(\nu + 1)} \int_a^t (t + \nu h - \sigma(s))_h^{(\nu-1)} f(s) \Delta s, \\ {}_h\Delta_b^{-\nu} f(t) &= h^\nu f(t - \nu h) + \frac{\nu}{\Gamma(\nu + 1)} \int_{\sigma(t)}^{\sigma(b)} (s + \nu h - \sigma(t))_h^{(\nu-1)} f(s) \Delta s. \end{aligned} \quad (4.6)$$

Theorem 79. *Let $f \in \mathcal{F}_{\mathbb{T}}$ and $\nu \geq 0$. For all $t \in \mathbb{T}^\kappa$ we have*

$${}_a\Delta_h^{-\nu} f^\Delta(t + \nu h) = ({}_a\Delta_h^{-\nu} f(t + \nu h))^\Delta - \frac{\nu}{\Gamma(\nu + 1)} (t + \nu h - a)_h^{(\nu-1)} f(a). \quad (4.7)$$

To prove Theorem 79 we make use of a technical lemma:

Lemma 80. *Let $t \in \mathbb{T}^\kappa$. The following equality holds for all $s \in \mathbb{T}^\kappa$:*

$$\begin{aligned} \Delta_{s,h} \left((t + \nu h - s)_h^{(\nu-1)} f(s) \right) \\ = (t + \nu h - \sigma(s))_h^{(\nu-1)} f^\Delta(s) - (\nu - 1)(t + \nu h - \sigma(s))_h^{(\nu-2)} f(s). \end{aligned} \quad (4.8)$$

Proof. Direct calculations give the intended result:

$$\begin{aligned} &\Delta_{s,h} \left((t + \nu h - s)_h^{(\nu-1)} f(s) \right) \\ &= \Delta_{s,h} \left((t + \nu h - s)_h^{(\nu-1)} \right) f(s) + (t + \nu h - \sigma(s))_h^{(\nu-1)} f^\Delta(s) \\ &= \frac{f(s)}{h} \left[h^{\nu-1} \frac{\Gamma \left(\frac{t + \nu h - \sigma(s)}{h} + 1 \right)}{\Gamma \left(\frac{t + \nu h - \sigma(s)}{h} + 1 - (\nu - 1) \right)} - h^{\nu-1} \frac{\Gamma \left(\frac{t + \nu h - s}{h} + 1 \right)}{\Gamma \left(\frac{t + \nu h - s}{h} + 1 - (\nu - 1) \right)} \right] \\ &\quad + (t + \nu h - \sigma(s))_h^{(\nu-1)} f^\Delta(s) \\ &= f(s) \left[h^{\nu-2} \left[\frac{\Gamma \left(\frac{t + \nu h - s}{h} \right)}{\Gamma \left(\frac{t - s}{h} + 1 \right)} - \frac{\Gamma \left(\frac{t + \nu h - s}{h} + 1 \right)}{\Gamma \left(\frac{t - s}{h} + 2 \right)} \right] \right] + (t + \nu h - \sigma(s))_h^{(\nu-1)} f^\Delta(s) \\ &= f(s) h^{\nu-2} \frac{\Gamma \left(\frac{t + \nu h - s - h}{h} + 1 \right)}{\Gamma \left(\frac{t - s + \nu h - h}{h} + 1 - (\nu - 2) \right)} (-\nu + 1) + (t + \nu h - \sigma(s))_h^{(\nu-1)} f^\Delta(s) \\ &= -(\nu - 1)(t + \nu h - \sigma(s))_h^{(\nu-2)} f(s) + (t + \nu h - \sigma(s))_h^{(\nu-1)} f^\Delta(s), \end{aligned}$$

where the first equality follows directly from (2.3). \square

Remark 81. Given an arbitrary $t \in \mathbb{T}^\kappa$ it is easy to prove, in a similar way as in the proof of Lemma 80, the following equality analogous to (4.8): for all $s \in \mathbb{T}^\kappa$

$$\begin{aligned} \Delta_{s,h} \left((s + \nu h - \sigma(t))_h^{(\nu-1)} f(s) \right) \\ = (\nu - 1)(s + \nu h - \sigma(t))_h^{(\nu-2)} f^\sigma(s) + (s + \nu h - \sigma(t))_h^{(\nu-1)} f^\Delta(s). \end{aligned} \quad (4.9)$$

Proof. (of Theorem 79) From Lemma 80 we obtain that

$$\begin{aligned}
 {}_a\Delta_h^{-\nu} f^\Delta(t + \nu h) &= h^\nu f^\Delta(t) + \frac{\nu}{\Gamma(\nu + 1)} \int_a^t (t + \nu h - \sigma(s))_h^{(\nu-1)} f^\Delta(s) \Delta s \\
 &= h^\nu f^\Delta(t) + \frac{\nu}{\Gamma(\nu + 1)} \left[(t + \nu h - s)_h^{(\nu-1)} f(s) \right]_{s=a}^{s=t} \\
 &\quad + \frac{\nu}{\Gamma(\nu + 1)} \int_a^{\sigma(t)} (\nu - 1)(t + \nu h - \sigma(s))_h^{(\nu-2)} f(s) \Delta s \\
 &= -\frac{\nu(t + \nu h - a)_h^{(\nu-1)}}{\Gamma(\nu + 1)} f(a) + h^\nu f^\Delta(t) + \nu h^{\nu-1} f(t) \\
 &\quad + \frac{\nu}{\Gamma(\nu + 1)} \int_a^t (\nu - 1)(t + \nu h - \sigma(s))_h^{(\nu-2)} f(s) \Delta s.
 \end{aligned} \tag{4.10}$$

We now show that $({}_a\Delta_h^{-\nu} f(t + \nu h))^\Delta$ equals (4.10):

$$\begin{aligned}
 ({}_a\Delta_h^{-\nu} f(t + \nu h))^\Delta &= \frac{1}{h} \left[h^\nu f(\sigma(t)) + \frac{\nu}{\Gamma(\nu + 1)} \int_a^{\sigma(t)} (\sigma(t) + \nu h - \sigma(s))_h^{(\nu-1)} f(s) \Delta s \right. \\
 &\quad \left. - h^\nu f(t) - \frac{\nu}{\Gamma(\nu + 1)} \int_a^t (t + \nu h - \sigma(s))_h^{(\nu-1)} f(s) \Delta s \right] \\
 &= h^\nu f^\Delta(t) + \frac{\nu}{h\Gamma(\nu + 1)} \left[\int_a^t (\sigma(t) + \nu h - \sigma(s))_h^{(\nu-1)} f(s) \Delta s \right. \\
 &\quad \left. - \int_a^t (t + \nu h - \sigma(s))_h^{(\nu-1)} f(s) \Delta s \right] + h^{\nu-1} \nu f(t) \\
 &= h^\nu f^\Delta(t) + \frac{\nu}{\Gamma(\nu + 1)} \int_a^t \Delta_{t,h} \left((t + \nu h - \sigma(s))_h^{(\nu-1)} \right) f(s) \Delta s + h^{\nu-1} \nu f(t) \\
 &= h^\nu f^\Delta(t) + \frac{\nu}{\Gamma(\nu + 1)} \int_a^t (\nu - 1)(t + \nu h - \sigma(s))_h^{(\nu-2)} f(s) \Delta s + \nu h^{\nu-1} f(t).
 \end{aligned}$$

□

Follows the counterpart of Theorem 79 for the right fractional h -sum:

Theorem 82. *Let $f \in \mathcal{F}_{\mathbb{T}}$ and $\nu \geq 0$. For all $t \in \mathbb{T}^\kappa$ we have*

$${}_h\Delta_{\rho(b)}^{-\nu} f^\Delta(t - \nu h) = \frac{\nu}{\Gamma(\nu + 1)} (b + \nu h - \sigma(t))_h^{(\nu-1)} f(b) + ({}_h\Delta_b^{-\nu} f(t - \nu h))^\Delta. \tag{4.11}$$

Proof. From (4.9) we obtain from integration by parts (item 2 of Lemma 27) that

$$\begin{aligned}
 {}_h\Delta_{\rho(b)}^{-\nu} f^\Delta(t - \nu h) &= \frac{\nu(b + \nu h - \sigma(t))_h^{(\nu-1)}}{\Gamma(\nu + 1)} f(b) + h^\nu f^\Delta(t) - \nu h^{\nu-1} f(\sigma(t)) \\
 &\quad - \frac{\nu}{\Gamma(\nu + 1)} \int_{\sigma(t)}^b (\nu - 1)(s + \nu h - \sigma(t))_h^{(\nu-2)} f^\sigma(s) \Delta s.
 \end{aligned} \tag{4.12}$$

We show that $({}_h\Delta_b^{-\nu} f(t - \nu h))^\Delta$ equals (4.12):

$$\begin{aligned}
& ({}_h\Delta_b^{-\nu} f(t - \nu h))^\Delta \\
&= h^\nu f^\Delta(t) + \frac{\nu}{h\Gamma(\nu + 1)} \left[\int_{\sigma^2(t)}^{\sigma(b)} (s + \nu h - \sigma^2(t))_h^{(\nu-1)} f(s) \Delta s \right. \\
&\quad \left. - \int_{\sigma^2(t)}^{\sigma(b)} (s + \nu h - \sigma(t))_h^{(\nu-1)} f(s) \Delta s \right] - \nu h^{\nu-1} f(\sigma(t)) \\
&= h^\nu f^\Delta(t) + \frac{\nu}{\Gamma(\nu + 1)} \int_{\sigma^2(t)}^{\sigma(b)} \Delta_{t,h} \left((s + \nu h - \sigma(t))_h^{(\nu-1)} \right) f(s) \Delta s - \nu h^{\nu-1} f(\sigma(t)) \\
&= h^\nu f^\Delta(t) - \frac{\nu}{\Gamma(\nu + 1)} \int_{\sigma^2(t)}^{\sigma(b)} (\nu - 1)(s + \nu h - \sigma^2(t))_h^{(\nu-2)} f(s) \Delta s - \nu h^{\nu-1} f(\sigma(t)) \\
&= h^\nu f^\Delta(t) - \frac{\nu}{\Gamma(\nu + 1)} \int_{\sigma(t)}^b (\nu - 1)(s + \nu h - \sigma(t))_h^{(\nu-2)} f(s) \Delta s - \nu h^{\nu-1} f(\sigma(t)).
\end{aligned}$$

□

Definition 83. Let $0 < \alpha \leq 1$ and set $\gamma := 1 - \alpha$. The *left fractional difference* ${}_a\Delta_h^\alpha f(t)$ and the *right fractional difference* ${}_h\Delta_b^\alpha f(t)$ of order α of a function $f \in \mathcal{F}_\mathbb{T}$ are defined as

$${}_a\Delta_h^\alpha f(t) := ({}_a\Delta_h^{-\gamma} f(t + \gamma h))^\Delta \quad \text{and} \quad {}_h\Delta_b^\alpha f(t) := -({}_h\Delta_b^{-\gamma} f(t - \gamma h))^\Delta$$

for all $t \in \mathbb{T}^\kappa$.

4.3 Main Results

Our aim is to introduce the h -fractional problem of the calculus of variations and to prove corresponding necessary optimality conditions. In order to obtain an Euler-Lagrange type equation (cf. Theorem 88) we first prove a fractional formula of h -summation by parts.

4.3.1 Fractional h -summation by parts

A big challenge was to discover a fractional h -summation by parts formula within the time scale setting. Indeed, there is no clue of what such a formula should be. We found it eventually, making use of the following lemma.

Lemma 84. *Let f and k be two functions defined on \mathbb{T}^κ and \mathbb{T}^{κ^2} , respectively, and g a function defined on $\mathbb{T}^\kappa \times \mathbb{T}^{\kappa^2}$. The following equality holds:*

$$\int_a^b f(t) \left[\int_a^t g(t, s)k(s)\Delta s \right] \Delta t = \int_a^{\rho(b)} k(t) \left[\int_{\sigma(t)}^b g(s, t)f(s)\Delta s \right] \Delta t.$$

Proof. Consider the matrices $R = [f(a+h), f(a+2h), \dots, f(b-h)]$,

$$C_1 = \begin{bmatrix} g(a+h, a)k(a) \\ g(a+2h, a)k(a) + g(a+2h, a+h)k(a+h) \\ \vdots \\ g(b-h, a)k(a) + g(b-h, a+h)k(a+h) + \dots + g(b-h, b-2h)k(b-2h) \end{bmatrix},$$

$$C_2 = \begin{bmatrix} g(a+h, a) \\ g(a+2h, a) \\ \vdots \\ g(b-h, a) \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0 \\ g(a+2h, a+h) \\ \vdots \\ g(b-h, a+h) \end{bmatrix}, \quad C_k = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ g(b-h, b-2h) \end{bmatrix}.$$

Direct calculations show that

$$\begin{aligned} \int_a^b f(t) \left[\int_a^t g(t, s)k(s)\Delta s \right] \Delta t &= h^2 \sum_{i=a/h}^{b/h-1} f(ih) \sum_{j=a/h}^{i-1} g(ih, jh)k(jh) = h^2 R \cdot C_1 \\ &= h^2 R \cdot [k(a)C_2 + k(a+h)C_3 + \dots + k(b-2h)C_k] \\ &= h^2 \left[k(a) \sum_{j=a/h+1}^{b/h-1} g(jh, a)f(jh) + k(a+h) \sum_{j=a/h+2}^{b/h-1} g(jh, a+h)f(jh) \right. \\ &\quad \left. + \dots + k(b-2h) \sum_{j=b/h-1}^{b/h-1} g(jh, b-2h)f(jh) \right] \\ &= \sum_{i=a/h}^{b/h-2} k(ih)h \sum_{j=\sigma(ih)/h}^{b/h-1} g(jh, ih)f(jh)h = \int_a^{\rho(b)} k(t) \left[\int_{\sigma(t)}^b g(s, t)f(s)\Delta s \right] \Delta t. \end{aligned}$$

□

Theorem 85 (fractional h -summation by parts). *Let f and g be real valued functions defined on \mathbb{T}^κ and \mathbb{T} , respectively. Fix $0 < \alpha \leq 1$ and put $\gamma := 1 - \alpha$. Then,*

$$\begin{aligned} \int_a^b f(t)_a \Delta_h^\alpha g(t) \Delta t &= h^\gamma f(\rho(b))g(b) - h^\gamma f(a)g(a) + \int_a^{\rho(b)} {}_h \Delta_{\rho(b)}^\alpha f(t)g^\sigma(t) \Delta t \\ &+ \frac{\gamma}{\Gamma(\gamma+1)} g(a) \left(\int_a^b (t+\gamma h-a)_h^{(\gamma-1)} f(t) \Delta t - \int_{\sigma(a)}^b (t+\gamma h-\sigma(a))_h^{(\gamma-1)} f(t) \Delta t \right). \end{aligned} \quad (4.13)$$

Proof. By (4.7) we can write

$$\begin{aligned}
\int_a^b f(t) {}_a\Delta_h^\alpha g(t) \Delta t &= \int_a^b f(t) ({}_a\Delta_h^{-\gamma} g(t + \gamma h))^\Delta \Delta t \\
&= \int_a^b f(t) \left[{}_a\Delta_h^{-\gamma} g^\Delta(t + \gamma h) + \frac{\gamma}{\Gamma(\gamma + 1)} (t + \gamma h - a)_h^{(\gamma-1)} g(a) \right] \Delta t \\
&= \int_a^b f(t) {}_a\Delta_h^{-\gamma} g^\Delta(t + \gamma h) \Delta t + \int_a^b \frac{\gamma}{\Gamma(\gamma + 1)} (t + \gamma h - a)_h^{(\gamma-1)} f(t) g(a) \Delta t.
\end{aligned} \tag{4.14}$$

Using (4.6) we get

$$\begin{aligned}
&\int_a^b f(t) {}_a\Delta_h^{-\gamma} g^\Delta(t + \gamma h) \Delta t \\
&= \int_a^b f(t) \left[h^\gamma g^\Delta(t) + \frac{\gamma}{\Gamma(\gamma + 1)} \int_a^t (t + \gamma h - \sigma(s))_h^{(\gamma-1)} g^\Delta(s) \Delta s \right] \Delta t \\
&= h^\gamma \int_a^b f(t) g^\Delta(t) \Delta t + \frac{\gamma}{\Gamma(\gamma + 1)} \int_a^{\rho(b)} g^\Delta(t) \int_{\sigma(t)}^b (s + \gamma h - \sigma(t))_h^{(\gamma-1)} f(s) \Delta s \Delta t \\
&= h^\gamma f(\rho(b)) [g(b) - g(\rho(b))] + \int_a^{\rho(b)} g^\Delta(t) {}_h\Delta_{\rho(b)}^{-\gamma} f(t - \gamma h) \Delta t,
\end{aligned}$$

where the third equality follows by Lemma 84. We proceed to develop the right hand side of the last equality as follows:

$$\begin{aligned}
&h^\gamma f(\rho(b)) [g(b) - g(\rho(b))] + \int_a^{\rho(b)} g^\Delta(t) {}_h\Delta_{\rho(b)}^{-\gamma} f(t - \gamma h) \Delta t \\
&= h^\gamma f(\rho(b)) [g(b) - g(\rho(b))] + \left[g(t) {}_h\Delta_{\rho(b)}^{-\gamma} f(t - \gamma h) \right]_{t=a}^{t=\rho(b)} \\
&\quad - \int_a^{\rho(b)} g^\sigma(t) ({}_h\Delta_{\rho(b)}^{-\gamma} f(t - \gamma h))^\Delta \Delta t \\
&= h^\gamma f(\rho(b)) g(b) - h^\gamma f(a) g(a) \\
&\quad - \frac{\gamma}{\Gamma(\gamma + 1)} g(a) \int_{\sigma(a)}^b (s + \gamma h - \sigma(a))_h^{(\gamma-1)} f(s) \Delta s + \int_a^{\rho(b)} ({}_h\Delta_{\rho(b)}^\alpha f(t)) g^\sigma(t) \Delta t,
\end{aligned}$$

where the first equality follows from Lemma 27. Putting this into (4.14) we get (4.13). \square

4.3.2 Necessary optimality conditions

We begin to fix two arbitrary real numbers α and β such that $\alpha, \beta \in (0, 1]$. Further, we put $\gamma := 1 - \alpha$ and $\nu := 1 - \beta$.

Let a function $L(t, u, v, w) : \mathbb{T}^\kappa \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be given. We consider the problem of minimizing (or maximizing) a functional $\mathcal{L} : \mathcal{F}_{\mathbb{T}} \rightarrow \mathbb{R}$ subject to given boundary conditions:

$$\mathcal{L}(y(\cdot)) = \int_a^b L(t, y^\sigma(t), {}_a\Delta_h^\alpha y(t), {}_h\Delta_b^\beta y(t)) \Delta t \longrightarrow \min, \quad y(a) = A, \quad y(b) = B. \quad (4.15)$$

Our main aim is to derive necessary optimality conditions for problem (4.15).

Definition 86. For $f \in \mathcal{F}_{\mathbb{T}}$ we define the norm

$$\|f\| = \max_{t \in \mathbb{T}^\kappa} |f^\sigma(t)| + \max_{t \in \mathbb{T}^\kappa} |{}_a\Delta_h^\alpha f(t)| + \max_{t \in \mathbb{T}^\kappa} |{}_h\Delta_b^\beta f(t)|.$$

A function $\hat{y} \in \mathcal{F}_{\mathbb{T}}$ with $\hat{y}(a) = A$ and $\hat{y}(b) = B$ is called a local minimizer for problem (4.15) provided there exists $\delta > 0$ such that $\mathcal{L}(\hat{y}) \leq \mathcal{L}(y)$ for all $y \in \mathcal{F}_{\mathbb{T}}$ with $y(a) = A$ and $y(b) = B$ and $\|y - \hat{y}\| < \delta$.

Definition 87. A function $\eta \in \mathcal{F}_{\mathbb{T}}$ is called an admissible variation provided $\eta \neq 0$ and $\eta(a) = \eta(b) = 0$.

From now on we assume that the second-order partial derivatives L_{uu} , L_{uv} , L_{uw} , L_{vw} , L_{vv} , and L_{ww} exist and are continuous.

First order optimality condition

Next theorem gives a first order necessary condition for problem (4.15), i.e., an Euler-Lagrange type equation for the fractional h -difference setting.

Theorem 88 (The h -fractional Euler-Lagrange equation for problem (4.15)). *If $\hat{y} \in \mathcal{F}_{\mathbb{T}}$ is a local minimizer for problem (4.15), then the equality*

$$L_u[\hat{y}](t) + {}_h\Delta_{\rho(b)}^\alpha L_v[\hat{y}](t) + {}_a\Delta_h^\beta L_w[\hat{y}](t) = 0 \quad (4.16)$$

holds for all $t \in \mathbb{T}^{\kappa^2}$ with operator $[\cdot]$ defined by $[y](s) = (s, y^\sigma(s), {}_a\Delta_s^\alpha y(s), {}_s\Delta_b^\beta y(s))$.

Proof. Suppose that $\hat{y}(\cdot)$ is a local minimizer of $\mathcal{L}[\cdot]$. Let $\eta(\cdot)$ be an arbitrarily fixed admissible variation and define a function $\Phi : \left(-\frac{\delta}{\|\eta(\cdot)\|}, \frac{\delta}{\|\eta(\cdot)\|}\right) \rightarrow \mathbb{R}$ by

$$\Phi(\varepsilon) = \mathcal{L}[\hat{y}(\cdot) + \varepsilon\eta(\cdot)]. \quad (4.17)$$

This function has a minimum at $\varepsilon = 0$, so we must have $\Phi'(0) = 0$, i.e.,

$$\int_a^b \left[L_u[\hat{y}](t)\eta^\sigma(t) + L_v[\hat{y}](t) {}_a\Delta_h^\alpha \eta(t) + L_w[\hat{y}](t) {}_h\Delta_b^\beta \eta(t) \right] \Delta t = 0,$$

which we may write equivalently as

$$\begin{aligned}
 hL_u[\hat{y}](t)\eta^\sigma(t)|_{t=\rho(b)} + \int_a^{\rho(b)} L_u[\hat{y}](t)\eta^\sigma(t)\Delta t + \int_a^b L_v[\hat{y}](t)_a\Delta_h^\alpha\eta(t)\Delta t \\
 + \int_a^b L_w[\hat{y}](t)_h\Delta_b^\beta\eta(t)\Delta t = 0. \quad (4.18)
 \end{aligned}$$

Using Theorem 85 and the fact that $\eta(a) = \eta(b) = 0$, we get

$$\int_a^b L_v[\hat{y}](t)_a\Delta_h^\alpha\eta(t)\Delta t = \int_a^{\rho(b)} ({}_h\Delta_{\rho(b)}^\alpha (L_v[\hat{y}])(t)) \eta^\sigma(t)\Delta t \quad (4.19)$$

for the third term in (4.18). Using (4.11) it follows that

$$\begin{aligned}
 & \int_a^b L_w[\hat{y}](t)_h\Delta_b^\beta\eta(t)\Delta t \\
 &= - \int_a^b L_w[\hat{y}](t)({}_h\Delta_b^{-\nu}\eta(t-\nu h))^\Delta\Delta t \\
 &= - \int_a^b L_w[\hat{y}](t) \left[{}_h\Delta_{\rho(b)}^{-\nu}\eta^\Delta(t-\nu h) - \frac{\nu}{\Gamma(\nu+1)}(b+\nu h-\sigma(t))_h^{(\nu-1)}\eta(b) \right] \Delta t \\
 &= - \int_a^b L_w[\hat{y}](t)_h\Delta_{\rho(b)}^{-\nu}\eta^\Delta(t-\nu h)\Delta t + \frac{\nu\eta(b)}{\Gamma(\nu+1)} \int_a^b (b+\nu h-\sigma(t))_h^{(\nu-1)}L_w[\hat{y}](t)\Delta t.
 \end{aligned} \quad (4.20)$$

We now use Lemma 84 to get

$$\begin{aligned}
 & \int_a^b L_w[\hat{y}](t)_h\Delta_{\rho(b)}^{-\nu}\eta^\Delta(t-\nu h)\Delta t \\
 &= \int_a^b L_w[\hat{y}](t) [h^\nu\eta^\Delta(t) \\
 & \quad + \frac{\nu}{\Gamma(\nu+1)} \int_{\sigma(t)}^b (s+\nu h-\sigma(t))_h^{(\nu-1)}\eta^\Delta(s)\Delta s] \Delta t \\
 &= \int_a^b h^\nu L_w[\hat{y}](t)\eta^\Delta(t)\Delta t \\
 & \quad + \frac{\nu}{\Gamma(\nu+1)} \int_a^{\rho(b)} \left[L_w[\hat{y}](t) \int_{\sigma(t)}^b (s+\nu h-\sigma(t))_h^{(\nu-1)}\eta^\Delta(s)\Delta s \right] \Delta t \\
 &= \int_a^b h^\nu L_w[\hat{y}](t)\eta^\Delta(t)\Delta t \\
 & \quad + \frac{\nu}{\Gamma(\nu+1)} \int_a^b \left[\eta^\Delta(t) \int_a^t (t+\nu h-\sigma(s))_h^{(\nu-1)}L_w[\hat{y}](s)\Delta s \right] \Delta t \\
 &= \int_a^b \eta^\Delta(t)_a\Delta_h^{-\nu} (L_w[\hat{y}])(t+\nu h)\Delta t.
 \end{aligned} \quad (4.21)$$

We apply again the time scale integration by parts formula (Lemma 27), this time to (4.21), to obtain,

$$\begin{aligned}
 & \int_a^b \eta^\Delta(t) {}_a\Delta_h^{-\nu} (L_w[\hat{y}]) (t + \nu h) \Delta t \\
 &= \int_a^{\rho(b)} \eta^\Delta(t) {}_a\Delta_h^{-\nu} (L_w[\hat{y}]) (t + \nu h) \Delta t \\
 &\quad + (\eta(b) - \eta(\rho(b))) {}_a\Delta_h^{-\nu} (L_w[\hat{y}]) (t + \nu h)|_{t=\rho(b)} \\
 &= [\eta(t) {}_a\Delta_h^{-\nu} (L_w[\hat{y}]) (t + \nu h)]_{t=a}^{t=\rho(b)} - \int_a^{\rho(b)} \eta^\sigma(t) ({}_a\Delta_h^{-\nu} (L_w[\hat{y}]) (t + \nu h))^\Delta \Delta t \\
 &\quad + \eta(b) {}_a\Delta_h^{-\nu} (L_w[\hat{y}]) (t + \nu h)|_{t=\rho(b)} - \eta(\rho(b)) {}_a\Delta_h^{-\nu} (L_w[\hat{y}]) (t + \nu h)|_{t=\rho(b)} \\
 &= \eta(b) {}_a\Delta_h^{-\nu} (L_w[\hat{y}]) (t + \nu h)|_{t=\rho(b)} - \eta(a) {}_a\Delta_h^{-\nu} (L_w[\hat{y}]) (t + \nu h)|_{t=a} \\
 &\quad - \int_a^{\rho(b)} \eta^\sigma(t) {}_a\Delta_h^\beta (L_w[\hat{y}]) (t) \Delta t.
 \end{aligned} \tag{4.22}$$

Since $\eta(a) = \eta(b) = 0$ we obtain, from (4.21) and (4.22), that

$$\int_a^b L_w[\hat{y}](t) {}_h\Delta_{\rho(b)}^{-\nu} \eta^\Delta(t) \Delta t = - \int_a^{\rho(b)} \eta^\sigma(t) {}_a\Delta_h^\beta (L_w[\hat{y}]) (t) \Delta t,$$

and after inserting in (4.20), that

$$\int_a^b L_w[\hat{y}](t) {}_h\Delta_b^\beta \eta(t) \Delta t = \int_a^{\rho(b)} \eta^\sigma(t) {}_a\Delta_h^\beta (L_w[\hat{y}]) (t) \Delta t. \tag{4.23}$$

By (4.19) and (4.23) we may write (4.18) as

$$\int_a^{\rho(b)} \left[L_u[\hat{y}](t) + {}_h\Delta_{\rho(b)}^\alpha (L_v[\hat{y}]) (t) + {}_a\Delta_h^\beta (L_w[\hat{y}]) (t) \right] \eta^\sigma(t) \Delta t = 0.$$

Since the values of $\eta^\sigma(t)$ are arbitrary for $t \in \mathbb{T}^{\kappa^2}$, the Euler-Lagrange equation (4.16) holds along \hat{y} . \square

The next result is a direct corollary of Theorem 88.

Corollary 89 (The h -Euler-Lagrange equation – cf., e.g., [37, 52]). *Let \mathbb{T} be the time scale $h\mathbb{Z}$, $h > 0$, with the forward jump operator σ and the delta derivative Δ . Assume $a, b \in \mathbb{T}$, $a < b$. If \hat{y} is a solution to the problem*

$$\mathcal{L}(y(\cdot)) = \int_a^b L(t, y^\sigma(t), y^\Delta(t)) \Delta t \longrightarrow \min, \quad y(a) = A, \quad y(b) = B,$$

then the equality $L_u(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)) - (L_v(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)))^\Delta = 0$ holds for all $t \in \mathbb{T}^{\kappa^2}$.

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Proof. Choose $\alpha = 1$ and a L that does not depend on w in Theorem 88. □

Remark 90. If we take $h = 1$ in Corollary 89 we have that

$$L_u(t, \hat{y}^\sigma(t), \Delta \hat{y}(t)) - \Delta L_v(t, \hat{y}^\sigma(t), \Delta \hat{y}(t)) = 0$$

holds for all $t \in \mathbb{T}^{\kappa^2}$. This equation is usually called *the discrete Euler-Lagrange equation*, and can be found, e.g., in [71, Chap. 8].

Natural boundary conditions

If the initial condition $y(a) = A$ is not present in problem (4.15) (i.e., $y(a)$ is free), besides the h -fractional Euler-Lagrange equation (4.16) the following supplementary condition must be fulfilled:

$$\begin{aligned} -h^\gamma L_v[\hat{y}](a) + \frac{\gamma}{\Gamma(\gamma+1)} & \left(\int_a^b (t + \gamma h - a)_h^{(\gamma-1)} L_v[\hat{y}](t) \Delta t \right. \\ & \left. - \int_{\sigma(a)}^b (t + \gamma h - \sigma(a))_h^{(\gamma-1)} L_v[\hat{y}](t) \Delta t \right) + L_w[\hat{y}](a) = 0. \end{aligned} \quad (4.24)$$

Similarly, if $y(b) = B$ is not present in (4.15) ($y(b)$ is free), the extra condition

$$\begin{aligned} hL_u[\hat{y}](\rho(b)) + h^\nu L_v[\hat{y}](\rho(b)) - h^\nu L_w[\hat{y}](\rho(b)) \\ + \frac{\nu}{\Gamma(\nu+1)} & \left(\int_a^b (b + \nu h - \sigma(t))_h^{(\nu-1)} L_w[\hat{y}](t) \Delta t \right. \\ & \left. - \int_a^{\rho(b)} (\rho(b) + \nu h - \sigma(t))_h^{(\nu-1)} L_w[\hat{y}](t) \Delta t \right) = 0 \end{aligned} \quad (4.25)$$

is added to Theorem 88. We leave the proof of the *natural boundary conditions* (4.24) and (4.25) to the reader. We just note here that the first term in (4.25) arises from the first term of the left hand side of (4.18).

Second order optimality condition

We now obtain a second order necessary condition for problem (4.15), i.e., we prove a Legendre optimality type condition for the fractional h -difference setting.

Theorem 91 (The h -fractional Legendre necessary condition). *If $\hat{y} \in \mathcal{F}_{\mathbb{T}}$ is a local minimizer for problem (4.15), then the inequality*

$$\begin{aligned}
 & h^2 L_{uu}[\hat{y}](t) + 2h^{\gamma+1} L_{uv}[\hat{y}](t) + 2h^{\nu+1}(\nu-1)L_{uw}[\hat{y}](t) + h^{2\gamma}(\gamma-1)^2 L_{vv}[\hat{y}](\sigma(t)) \\
 & + 2h^{\nu+\gamma}(\gamma-1)L_{vw}[\hat{y}](\sigma(t)) + 2h^{\nu+\gamma}(\nu-1)L_{vw}[\hat{y}](t) + h^{2\nu}(\nu-1)^2 L_{ww}[\hat{y}](t) \\
 & + h^{2\nu} L_{ww}[\hat{y}](\sigma(t)) + \int_a^t h^3 L_{ww}[\hat{y}](s) \left(\frac{\nu(1-\nu)}{\Gamma(\nu+1)} (t + \nu h - \sigma(s))_h^{(\nu-2)} \right)^2 \Delta s \\
 & + h^\gamma L_{vv}[\hat{y}](t) + \int_{\sigma(\sigma(t))}^b h^3 L_{vv}[\hat{y}](s) \left(\frac{\gamma(\gamma-1)}{\Gamma(\gamma+1)} (s + \gamma h - \sigma(\sigma(t)))_h^{(\gamma-2)} \right)^2 \Delta s \geq 0
 \end{aligned} \tag{4.26}$$

holds for all $t \in \mathbb{T}^{\kappa^2}$, where $[\hat{y}](t) = (t, \hat{y}^\sigma(t), {}_a\Delta_t^\alpha \hat{y}(t), {}_t\Delta_b^\beta \hat{y}(t))$.

Proof. By the hypothesis of the theorem, and letting Φ be as in (4.17), we have as necessary optimality condition that $\Phi''(0) \geq 0$ for an arbitrary admissible variation $\eta(\cdot)$. Inequality $\Phi''(0) \geq 0$ is equivalent to

$$\begin{aligned}
 & \int_a^b \left[L_{uu}[\hat{y}](t)(\eta^\sigma(t))^2 + 2L_{uv}[\hat{y}](t)\eta^\sigma(t) {}_a\Delta_h^\alpha \eta(t) + 2L_{uw}[\hat{y}](t)\eta^\sigma(t) {}_h\Delta_b^\beta \eta(t) \right. \\
 & \left. + L_{vv}[\hat{y}](t)({}_a\Delta_h^\alpha \eta(t))^2 + 2L_{vw}[\hat{y}](t) {}_a\Delta_h^\alpha \eta(t) {}_h\Delta_b^\beta \eta(t) + L_{ww}[\hat{y}](t)({}_h\Delta_b^\beta \eta(t))^2 \right] \Delta t \geq 0.
 \end{aligned} \tag{4.27}$$

Let $\tau \in \mathbb{T}^{\kappa^2}$ be arbitrary, and choose $\eta : \mathbb{T} \rightarrow \mathbb{R}$ given by $\eta(t) = \begin{cases} h & \text{if } t = \sigma(\tau); \\ 0 & \text{otherwise.} \end{cases}$ It follows that $\eta(a) = \eta(b) = 0$, i.e., η is an admissible variation. Using (4.7) we get

$$\begin{aligned}
 & \int_a^b \left[L_{uu}[\hat{y}](t)(\eta^\sigma(t))^2 + 2L_{uv}[\hat{y}](t)\eta^\sigma(t) {}_a\Delta_h^\alpha \eta(t) + L_{vv}[\hat{y}](t)({}_a\Delta_h^\alpha \eta(t))^2 \right] \Delta t \\
 & = \int_a^b \left[L_{uu}[\hat{y}](t)(\eta^\sigma(t))^2 \right. \\
 & \quad + 2L_{uv}[\hat{y}](t)\eta^\sigma(t) \left(h^\gamma \eta^\Delta(t) + \frac{\gamma}{\Gamma(\gamma+1)} \int_a^t (t + \gamma h - \sigma(s))_h^{(\gamma-1)} \eta^\Delta(s) \Delta s \right) \\
 & \quad \left. + L_{vv}[\hat{y}](t) \left(h^\gamma \eta^\Delta(t) + \frac{\gamma}{\Gamma(\gamma+1)} \int_a^t (t + \gamma h - \sigma(s))_h^{(\gamma-1)} \eta^\Delta(s) \Delta s \right)^2 \right] \Delta t \\
 & = h^3 L_{uu}[\hat{y}](\tau) + 2h^{\gamma+2} L_{uv}[\hat{y}](\tau) + h^{\gamma+1} L_{vv}[\hat{y}](\tau) \\
 & \quad + \int_{\sigma(\tau)}^b L_{vv}[\hat{y}](t) \left(h^\gamma \eta^\Delta(t) + \frac{\gamma}{\Gamma(\gamma+1)} \int_a^t (t + \gamma h - \sigma(s))_h^{(\gamma-1)} \eta^\Delta(s) \Delta s \right)^2 \Delta t.
 \end{aligned}$$

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Observe that

$$\begin{aligned}
& h^{2\gamma+1}(\gamma-1)^2 L_{vv}[\hat{y}](\sigma(\tau)) \\
& + \int_{\sigma^2(\tau)}^b L_{vv}[\hat{y}](t) \left(\frac{\gamma}{\Gamma(\gamma+1)} \int_a^t (t+\gamma h - \sigma(s))_h^{(\gamma-1)} \eta^\Delta(s) \Delta s \right)^2 \Delta t \\
& = \int_{\sigma(\tau)}^b L_{vv}[\hat{y}](t) \left(h^\gamma \eta^\Delta(t) + \frac{\gamma}{\Gamma(\gamma+1)} \int_a^t (t+\gamma h - \sigma(s))_h^{(\gamma-1)} \eta^\Delta(s) \Delta s \right)^2 \Delta t.
\end{aligned}$$

Let $t \in [\sigma^2(\tau), \rho(b)] \cap h\mathbb{Z}$. Since

$$\begin{aligned}
& \frac{\gamma}{\Gamma(\gamma+1)} \int_a^t (t+\gamma h - \sigma(s))_h^{(\gamma-1)} \eta^\Delta(s) \Delta s \\
& = \frac{\gamma}{\Gamma(\gamma+1)} \left[\int_a^{\sigma(\tau)} (t+\gamma h - \sigma(s))_h^{(\gamma-1)} \eta^\Delta(s) \Delta s \right. \\
& \quad \left. + \int_{\sigma(\tau)}^t (t+\gamma h - \sigma(s))_h^{(\gamma-1)} \eta^\Delta(s) \Delta s \right] \\
& = h \frac{\gamma}{\Gamma(\gamma+1)} \left[(t+\gamma h - \sigma(\tau))_h^{(\gamma-1)} - (t+\gamma h - \sigma(\sigma(\tau)))_h^{(\gamma-1)} \right] \\
& = \frac{\gamma h^\gamma}{\Gamma(\gamma+1)} \left[\frac{(\frac{t-\tau}{h} + \gamma - 1) \Gamma(\frac{t-\tau}{h} + \gamma - 1) - (\frac{t-\tau}{h}) \Gamma(\frac{t-\tau}{h} + \gamma - 1)}{(\frac{t-\tau}{h}) \Gamma(\frac{t-\tau}{h})} \right] \\
& = h^2 \frac{\gamma(\gamma-1)}{\Gamma(\gamma+1)} (t+\gamma h - \sigma(\sigma(\tau)))_h^{(\gamma-2)},
\end{aligned} \tag{4.28}$$

we conclude that

$$\begin{aligned}
& \int_{\sigma^2(\tau)}^b L_{vv}[\hat{y}](t) \left(\frac{\gamma}{\Gamma(\gamma+1)} \int_a^t (t+\gamma h - \sigma(s))_h^{(\gamma-1)} \eta^\Delta(s) \Delta s \right)^2 \Delta t \\
& = \int_{\sigma^2(\tau)}^b L_{vv}[\hat{y}](t) \left(h^2 \frac{\gamma(\gamma-1)}{\Gamma(\gamma+1)} (t+\gamma h - \sigma^2(\tau))_h^{(\gamma-2)} \right)^2 \Delta t.
\end{aligned}$$

Note that we can write ${}_t\Delta_b^\beta \eta(t) = -{}_h\Delta_{\rho(b)}^{-\nu} \eta^\Delta(t - \nu h)$ because $\eta(b) = 0$. It is not difficult to see that the following equality holds:

$$\begin{aligned}
& \int_a^b 2L_{uv}[\hat{y}](t) \eta^\sigma(t) {}_h\Delta_b^\beta \eta(t) \Delta t = - \int_a^b 2L_{uv}[\hat{y}](t) \eta^\sigma(t) {}_h\Delta_{\rho(b)}^{-\nu} \eta^\Delta(t - \nu h) \Delta t \\
& = 2h^{2+\nu} L_{uv}[\hat{y}](\tau) (\nu - 1).
\end{aligned}$$

Moreover,

$$\begin{aligned}
 & \int_a^b 2L_{vw}[\hat{y}](t) {}_a\Delta_h^\alpha \eta(t) {}_h\Delta_b^\beta \eta(t) \Delta t \\
 &= -2 \int_a^b L_{vw}[\hat{y}](t) \left\{ \left(h^\gamma \eta^\Delta(t) + \frac{\gamma}{\Gamma(\gamma+1)} \cdot \int_a^t (t + \gamma h - \sigma(s))_h^{(\gamma-1)} \eta^\Delta(s) \Delta s \right) \right. \\
 &\quad \left. \cdot \left[h^\nu \eta^\Delta(t) + \frac{\nu}{\Gamma(\nu+1)} \int_{\sigma(t)}^b (s + \nu h - \sigma(t))_h^{(\nu-1)} \eta^\Delta(s) \Delta s \right] \right\} \Delta t \\
 &= 2h^{\gamma+\nu+1}(\nu-1)L_{vw}[\hat{y}](\tau) + 2h^{\gamma+\nu+1}(\gamma-1)L_{vw}[\hat{y}](\sigma(\tau)).
 \end{aligned}$$

Finally, we have that

$$\begin{aligned}
 & \int_a^b L_{ww}[\hat{y}](t) ({}_h\Delta_b^\beta \eta(t))^2 \Delta t \\
 &= \int_a^{\sigma(\sigma(\tau))} L_{ww}[\hat{y}](t) \left[h^\nu \eta^\Delta(t) + \frac{\nu}{\Gamma(\nu+1)} \int_{\sigma(t)}^b (s + \nu h - \sigma(t))_h^{(\nu-1)} \eta^\Delta(s) \Delta s \right]^2 \Delta t \\
 &= \int_a^\tau L_{ww}[\hat{y}](t) \left[\frac{\nu}{\Gamma(\nu+1)} \int_{\sigma(t)}^b (s + \nu h - \sigma(t))_h^{(\nu-1)} \eta^\Delta(s) \Delta s \right]^2 \Delta t \\
 &\quad + hL_{ww}[\hat{y}](\tau)(h^\nu - \nu h^\nu)^2 + h^{2\nu+1}L_{ww}[\hat{y}](\sigma(\tau)) \\
 &= \int_a^\tau L_{ww}[\hat{y}](t) \left[h \frac{\nu}{\Gamma(\nu+1)} \left\{ (\tau + \nu h - \sigma(t))_h^{(\nu-1)} - (\sigma(\tau) + \nu h - \sigma(t))_h^{(\nu-1)} \right\} \right]^2 \Delta t \\
 &\quad + hL_{ww}[\hat{y}](\tau)(h^\nu - \nu h^\nu)^2 + h^{2\nu+1}L_{ww}[\hat{y}](\sigma(\tau)).
 \end{aligned}$$

Similarly as we did in (4.28), we can prove that

$$\begin{aligned}
 & h \frac{\nu}{\Gamma(\nu+1)} \left\{ (\tau + \nu h - \sigma(t))_h^{(\nu-1)} - (\sigma(\tau) + \nu h - \sigma(t))_h^{(\nu-1)} \right\} \\
 &= h^2 \frac{\nu(1-\nu)}{\Gamma(\nu+1)} (\tau + \nu h - \sigma(t))_h^{(\nu-2)}.
 \end{aligned}$$

Thus, we have that inequality (4.27) is equivalent to

$$\begin{aligned}
 & h \left\{ h^2 L_{uu}[\hat{y}](t) + 2h^{\gamma+1} L_{uv}[\hat{y}](t) + h^\gamma L_{vv}[\hat{y}](t) + L_{vv}(\sigma(t))(\gamma h^\gamma - h^\gamma)^2 \right. \\
 &\quad + \int_{\sigma(\sigma(t))}^b h^3 L_{vv}(s) \left(\frac{\gamma(\gamma-1)}{\Gamma(\gamma+1)} (s + \gamma h - \sigma(\sigma(t)))_h^{(\gamma-2)} \right)^2 \Delta s \\
 &\quad + 2h^{\nu+1} L_{uw}[\hat{y}](t)(\nu-1) + 2h^{\gamma+\nu}(\nu-1)L_{vw}[\hat{y}](t) \\
 &\quad + 2h^{\gamma+\nu}(\gamma-1)L_{vw}(\sigma(t)) + h^{2\nu} L_{ww}[\hat{y}](t)(1-\nu)^2 + h^{2\nu} L_{ww}[\hat{y}](\sigma(t)) \\
 &\quad \left. + \int_a^t h^3 L_{ww}[\hat{y}](s) \left(\frac{\nu(1-\nu)}{\Gamma(\nu+1)} (t + \nu h - \sigma(s))^{\nu-2} \right)^2 \Delta s \right\} \geq 0. \quad (4.29)
 \end{aligned}$$

Because $h > 0$, (4.29) is equivalent to (4.26). The theorem is proved. \square

The next result is a simple corollary of Theorem 91.

Corollary 92 (The h -Legendre necessary condition – cf. Result 1.3 of [37]). *Let \mathbb{T} be the time scale $h\mathbb{Z}$, $h > 0$, with the forward jump operator σ and the delta derivative Δ . Assume $a, b \in \mathbb{T}$, $a < b$. If \hat{y} is a solution to the problem*

$$\mathcal{L}(y(\cdot)) = \int_a^b L(t, y^\sigma(t), y^\Delta(t)) \Delta t \longrightarrow \min, \quad y(a) = A, \quad y(b) = B,$$

then the inequality

$$h^2 L_{uu}[\hat{y}](t) + 2h L_{uv}[\hat{y}](t) + L_{vv}[\hat{y}](t) + L_{vv}[\hat{y}](\sigma(t)) \geq 0 \quad (4.30)$$

holds for all $t \in \mathbb{T}^{\kappa^2}$, where $[\hat{y}](t) = (t, \hat{y}^\sigma(t), \hat{y}^\Delta(t))$.

Proof. Choose $\alpha = 1$ and a Lagrangian L that does not depend on w . Then, $\gamma = 0$ and the result follows immediately from Theorem 91. \square

Remark 93. When h goes to zero we have $\sigma(t) = t$ and inequality (4.30) coincides with Legendre's classical necessary optimality condition $L_{vv}[\hat{y}](t) \geq 0$ (cf., e.g., [114]).

4.4 Examples

In this section we present some illustrative examples.

Example 94. Let us consider the following problem:

$$\mathcal{L}(y) = \frac{1}{2} \int_0^1 \left({}_0\Delta_h^{\frac{3}{4}} y(t) \right)^2 \Delta t \longrightarrow \min, \quad y(0) = 0, \quad y(1) = 1. \quad (4.31)$$

We consider (4.31) with different values of h . Numerical results show that when h tends to zero the h -fractional Euler-Lagrange extremal tends to the fractional continuous extremal: when $h \rightarrow 0$ (4.31) tends to the fractional continuous variational problem in the Riemann-Liouville sense studied in [3, Example 1], with solution given by

$$y(t) = \frac{1}{2} \int_0^t \frac{dx}{[(1-x)(t-x)]^{\frac{1}{4}}}. \quad (4.32)$$

This is illustrated in Figure 4.1.

In this example for each value of h there is a unique h -fractional Euler-Lagrange extremal, solution of (4.16), which always verifies the h -fractional Legendre necessary condition (4.26).

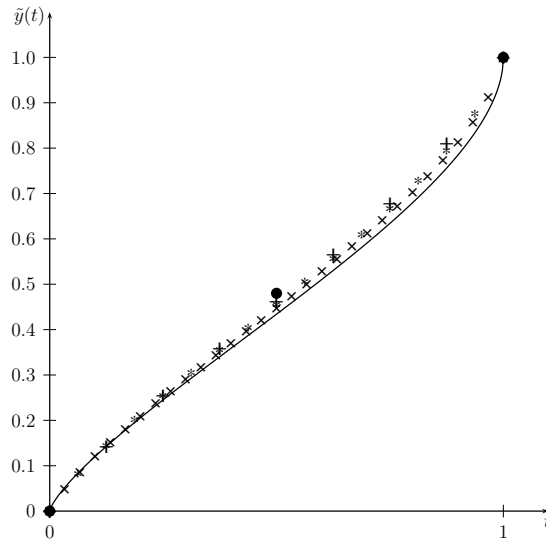


Figure 4.1: Extremal $\tilde{y}(t)$ for problem of Example 94 with different values of h : $h = 0.50$ (\bullet); $h = 0.125$ ($+$); $h = 0.0625$ ($*$); $h = 1/30$ (\times). The continuous line represent function (4.32).

Example 95. Let us consider the following problem:

$$\mathcal{L}(y) = \int_0^1 \left[\frac{1}{2} ({}_0\Delta_h^\alpha y(t))^2 - y^\sigma(t) \right] \Delta t \longrightarrow \min, \quad y(0) = 0, \quad y(1) = 0. \quad (4.33)$$

We begin by considering problem (4.33) with a fixed value for α and different values of h . The extremals \tilde{y} are obtained using our Euler-Lagrange equation (4.16). As in Example 94 the numerical results show that when h tends to zero the extremal of the problem tends to the extremal of the corresponding continuous fractional problem of the calculus of variations in the Riemann-Liouville sense. More precisely, when h approximates zero problem (4.33) tends to the fractional continuous problem studied in [6, Example 2]. For $\alpha = 1$ and $h \rightarrow 0$ the extremal of (4.33) is given by $y(t) = \frac{1}{2}t(1-t)$, which coincides with the extremal of the classical problem of the calculus of variations

$$\mathcal{L}(y) = \int_0^1 \left(\frac{1}{2}y'(t)^2 - y(t) \right) dt \longrightarrow \min, \quad y(0) = 0, \quad y(1) = 0.$$

This is illustrated in Figure 4.2 for $h = \frac{1}{2^i}$, $i = 1, 2, 3, 4$.

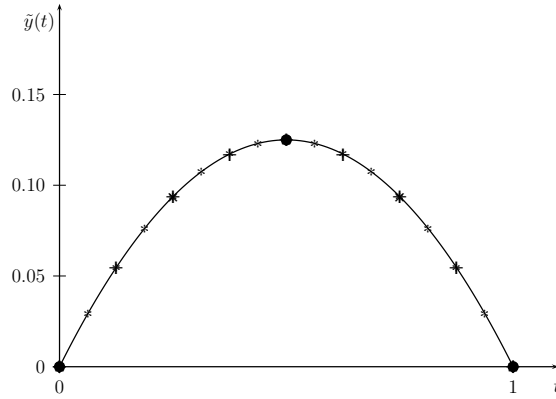


Figure 4.2: Extremal $\tilde{y}(t)$ for problem (4.33) with $\alpha = 1$ and different values of h : $h = 0.5$ (\bullet); $h = 0.25$ (\times); $h = 0.125$ ($+$); $h = 0.0625$ ($*$).

In this example, for each value of α and h , we only have one extremal (we only have one solution to (4.16) for each α and h). Our Legendre condition (4.26) is always verified along the extremals. Figure 4.3 shows the extremals of problem (4.33) for a fixed value of h ($h = 1/20$) and different values of α . The numerical results show that when α tends to one the extremal tends to the solution of the classical (integer order) discrete-time problem.

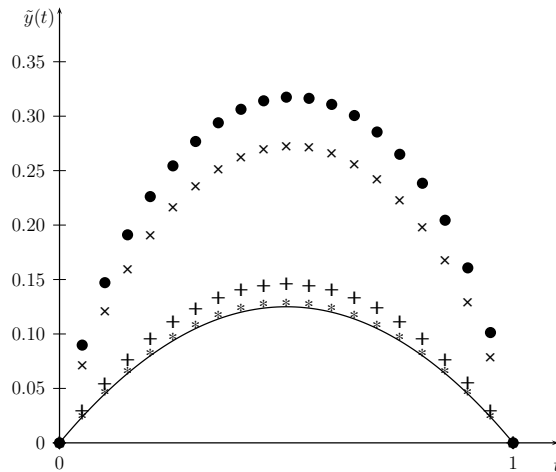


Figure 4.3: Extremal $\tilde{y}(t)$ for (4.33) with $h = 0.05$ and different values of α : $\alpha = 0.70$ (\bullet); $\alpha = 0.75$ (\times); $\alpha = 0.95$ ($+$); $\alpha = 0.99$ ($*$). The continuous line is $y(t) = \frac{1}{2}t(1 - t)$.

Our last example shows that the h -fractional Legendre necessary optimality condition can be a very useful tool. In Example 96 we consider a problem for which the h -fractional Euler-Lagrange equation gives several candidates but just a few of them verify the Legendre condition (4.26).

Example 96. Let us consider the following problem:

$$\mathcal{L}(y) = \int_a^b ({}_a\Delta_h^\alpha y(t))^3 + \theta ({}_h\Delta_b^\alpha y(t))^2 \Delta t \longrightarrow \min, \quad y(a) = 0, \quad y(b) = 1. \quad (4.34)$$

For $\alpha = 0.8$, $\beta = 0.5$, $h = 0.25$, $a = 0$, $b = 1$, and $\theta = 1$, problem (4.34) has eight different Euler-Lagrange extremals. As we can see on Table 4.1 only two of the candidates verify the Legendre condition. To determine the best candidate we compare the values of the functional \mathcal{L} along the two good candidates. The extremal we are looking for is given by the candidate number five on Table 4.1.

#	$\tilde{y}(\frac{1}{4})$	$\tilde{y}(\frac{1}{2})$	$\tilde{y}(\frac{3}{4})$	$\mathcal{L}(\tilde{y})$	Legendre condition (4.26)
1	-0.5511786	0.0515282	0.5133134	9.3035911	Not verified
2	0.2669091	0.4878808	0.7151924	2.0084203	Verified
3	-2.6745703	0.5599360	-2.6730125	698.4443232	Not verified
4	0.5789976	1.0701515	0.1840377	12.5174960	Not verified
5	1.0306820	1.8920322	2.7429222	-32.7189756	Verified
6	0.5087946	-0.1861431	0.4489196	10.6730959	Not verified
7	4.0583690	-1.0299054	-5.0030989	2451.7637948	Not verified
8	-1.7436106	-3.1898449	-0.8850511	238.6120299	Not verified

Table 4.1: There exist 8 Euler-Lagrange extremals for problem (4.34) with $\alpha = 0.8$, $\beta = 0.5$, $h = 0.25$, $a = 0$, $b = 1$, and $\theta = 1$, but only 2 of them satisfy the fractional Legendre condition (4.26).

For problem (4.34) with $\alpha = 0.3$, $h = 0.1$, $a = 0$, $b = 0.5$, and $\theta = 0$, we obtain the results of Table 4.2: there exist sixteen Euler-Lagrange extremals but only one satisfy the fractional Legendre condition. The extremal we are looking for is given by the candidate number six on Table 4.2.

4.4. EXAMPLES

#	$\tilde{y}(0.1)$	$\tilde{y}(0.2)$	$\tilde{y}(0.3)$	$\tilde{y}(0.4)$	$\mathcal{L}(\tilde{y})$	(4.26)
1	-0.305570704	-0.428093486	0.223708338	0.480549114	12.25396166	No
2	-0.427934654	-0.599520948	0.313290997	-0.661831134	156.2317667	No
3	0.284152257	-0.227595659	0.318847274	0.531827387	8.669645848	No
4	-0.277642565	0.222381632	0.386666793	0.555841555	6.993518478	No
5	0.387074742	-0.310032839	0.434336603	-0.482903047	110.7912605	No
6	0.259846344	0.364035314	0.463222456	0.597907505	5.104389191	Yes
7	-0.375094681	0.300437245	0.522386246	-0.419053781	93.95316858	No
8	0.343327771	0.480989769	0.61204299	-0.280908953	69.23497954	No
9	0.297792192	0.417196073	-0.218013689	0.460556635	14.12227593	No
10	0.41283304	0.578364133	-0.302235104	-0.649232892	157.8272685	No
11	-0.321401682	0.257431098	-0.360644857	0.400971272	19.87468886	No
12	0.330157414	-0.264444122	-0.459803086	0.368850105	24.84475504	No
13	-0.459640837	0.368155651	-0.515763025	-0.860276767	224.9964788	No
14	-0.359429958	-0.50354835	-0.640748011	0.294083676	34.43515839	No
15	0.477760586	-0.382668914	-0.66536683	-0.956478654	263.3075289	No
16	-0.541587541	-0.758744525	-0.965476394	-1.246195157	392.9592508	No

Table 4.2: There exist 16 Euler-Lagrange extremals for problem (4.34) with $\alpha = 0.3$, $h = 0.1$, $a = 0$, $b = 0.5$, and $\theta = 0$, but only 1 (candidate #6) satisfy the fractional Legendre condition (4.26).

4.5 Conclusion

In this chapter we introduce a new fractional difference variational calculus in the time-scale $(h\mathbb{Z})_a$, $h > 0$ and a a real number, for Lagrangians depending on left and right discrete-time fractional derivatives. Our objective was to introduce the concept of left and right fractional sum/difference (cf. Definition 76) and to develop the theory of fractional difference calculus. An Euler–Lagrange type equation (4.16), fractional natural boundary conditions (4.24) and (4.25), and a second order Legendre type necessary optimality condition (4.26), were obtained. The results are based on a new discrete fractional summation by parts formula (4.13) for $(h\mathbb{Z})_a$. Obtained first and second order necessary optimality conditions were implemented computationally in the computer algebra systems **Maple** and **Maxima** (the **Maxima** code is found in Appendix B). Our numerical results show that solutions to the considered fractional problems become the classical discrete-time solutions when the fractional order of the discrete-derivatives are integer values, and that they converge to the fractional continuous-time solutions when h tends to zero. Our Legendre type condition is useful to eliminate false candidates identified via the Euler-Lagrange fractional equation. The results of the chapter are formulated using standard notations of the theory of time scales [41, 69, 76] because we keep in our mind the desire to generalize the present results to an arbitrary time scale \mathbb{T} . Undoubtedly, much remains to be done in the development of the theory of discrete fractional calculus of variations in $(h\mathbb{Z})_a$ here initiated.

4.6 State of the Art

The results of this chapter are published in [34] and were presented by the author at the FSS’09, Symposium on Fractional Signals and Systems, Lisbon, Portugal, November 4–6, 2009, in a contributed talk entitled *The fractional difference calculus of variations*.

Recently, Ferreira and Torres [53] continued the development of the theory presented here with handy tools for the explicit solution of discrete equations involving left and right fractional difference operators.

Chapter 5

Fractional Derivatives and Integrals on arbitrary \mathbb{T}

In this chapter we introduce a fractional calculus on time scales using the theory of delta dynamic equations. The basic notions of fractional order integral and fractional order derivative on an arbitrary time scale are proposed, using the inverse Laplace transform on time scales. Useful properties of the new fractional operators are proved.

5.1 Introduction

Recently, two attempts have been made to provide a general definition of fractional derivative on an arbitrary time scale (see [18, 104]). These two works address a very interesting question, but unfortunately there is a small inconsistency in the very beginning of both studies. Indeed, investigations [18, 104] are based on the following definition of generalized polynomials on time scales $h_\alpha : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}$:

$$\begin{aligned} h_0(t, s) &= 1, \\ h_{\alpha+1}(t, s) &= \int_s^t h_\alpha(\tau, s) \Delta\tau. \end{aligned} \tag{5.1}$$

Although the recurrence formula above address a very interesting question about how to define fractional generalized polynomials on time scales, recursion (5.1) provides a definition only in the case $\alpha \in \mathbb{N}_0$, and there is no hope to define polynomials h_α for real or complex indices α with (5.1).

Here we propose a different approach to provide a general definition of fractional derivative on an arbitrary time scale based on the Laplace transform [42].

The chapter is organized as follows. In Section 5.2 we review the basic notions of Laplace transform on \mathbb{R} (Section 5.2.1) and we introduce some necessary tools from time scales (Section 5.2.2). Our results are then given in Section 5.3: we introduce the concept of fractional integral and fractional derivative on an arbitrary time scale \mathbb{T} (Section 5.3.1); we then prove some important properties of the fractional integrals and derivatives (Section 5.3.2).

5.2 Preliminaries

We start this section recalling results on classical Laplace transforms. All necessary concepts to understand the Laplace transform definition on time scales are supplied.

5.2.1 Laplace transform on \mathbb{R} as motivation

In order to motivate our idea we start with some concepts of continuous classical Laplace transform.

Definition 97. The *Laplace transform* of $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$\mathcal{L}[f](z) = F(z) = \int_0^{\infty} e^{-zt} f(t) dt, \quad (5.2)$$

where $z \in \mathbb{C}$ is chosen so that the integral converges absolutely.

Remark 98. It is well known that if

1. f is piecewise continuous on the interval $0 \leq t \leq A$ for any positive A ;
2. $|f(t)| \leq Me^{at}$ when $t \geq T$, for any real constant a and any positive constants M and T (this means that f is of exponential order, i.e., its rate of growth is not faster than that of exponential functions);

then the Laplace transform (5.2) exists for $z > a$.

Because of the usefulness of the Laplace transform of derivatives in our work, we now refer what happens in the continuous case, using the following theorem that can be easily proved using the definition of Laplace transform and integration by parts.

Theorem 99. *Suppose f is of exponential order, and that f is continuous and f' is piecewise continuous on any interval $0 \leq t \leq A$. Then*

$$\mathcal{L}[f'] = z\mathcal{L}[f](z) - f(0) . \quad (5.3)$$

Remark 100. Applying the theorem multiple times yield

$$\mathcal{L}[f^{(n)}](z) = z^n \mathcal{L}[f] - z^{n-1} f(0) - z^{n-2} f'(0) - \dots - z^2 f^{(n-3)}(0) - z f^{(n-2)}(0) - f^{(n-1)}(0). \quad (5.4)$$

The above equality is an extremely useful tool of the Laplace transform for solving linear ODEs with constant coefficients because it converts linear differential equations to linear algebraic equations that can be solved easily.

Next proposition gives the Laplace transform of the Caputo fractional derivative and was an inspiration for our work.

Proposition 101. [72] *Let $\alpha > 0$, n be the integer such that $n - 1 < \alpha \leq n$, and f a function satisfying $f \in C^n(\mathbb{R}^+)$, $f^{(n)} \in L_1(0, t_1)$, $t_1 > 0$, and $|f^{(n)}(t)| \leq Be^{q_0 t}$, $t > t_1 > 0$. If the Laplace transforms $\mathcal{L}[f](z)$ and $\mathcal{L}[f^{(n)}](z)$ exist, and $\lim_{t \rightarrow +\infty} f^{(k)}(t) = 0$ for $k = 0, \dots, n - 1$, then*

$$\mathcal{L} [{}_0^C D_x^\alpha f] (z) = z^\alpha \mathcal{L} [f] (z) - \sum_{k=0}^{n-1} f^{(k)}(0) z^{\alpha-k-1} .$$

Remark 102. If $\alpha \in (0, 1]$, then $\mathcal{L} [{}_0^C D_x^\alpha f] (z) = z^\alpha \mathcal{L} [f] (z) - f(0) z^{\alpha-1}$.

5.2.2 The Laplace transform on time scales

Looking to Definition 97 we can observe that, in classical calculus, the exponential function assumes an important role in the theory of Laplace transforms.

Following the same direction, we would like to have a function on time scales that serve the same purpose as the exponential function does in the real case.

Before we can define such function, we need several concepts in order for the definition to make sense. The first concept that we need is the concept of regressivity.

Definition 103. A function $p : \mathbb{T}^\kappa \rightarrow \mathbb{R}$ is *regressive* provided

$$1 + \mu(t)p(t) \neq 0$$

holds for all $t \in \mathbb{T}^\kappa$. We denote by \mathcal{R} the set of all regressive and rd-continuous functions. The set of all *positively regressive* functions is defined by

$$\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \text{ for all } t \in \mathbb{T}^\kappa\}.$$

Definition 104. The function $(\ominus p)(t) = -\frac{p(t)}{1+\mu(t)p(t)}$ for all $t \in \mathbb{T}^\kappa$ and $p \in \mathcal{R}$.

Now we define the exponential function on time scales (also called, in some literature, the generalized exponential function) as the solution of the IVP (5.5).

Theorem 105. [43, Theorem 1.37] Suppose $p \in \mathcal{R}$ and fix $t_0 \in \mathbb{T}$. Then the initial value problem

$$y^\Delta = p(t)y, \quad y(t_0) = 1 \tag{5.5}$$

has a unique solution on \mathbb{T} .

The solution of the IVP (5.5) is the *exponential function on time scales*, and is given by $e_p(\cdot, t_0)$, where

$$e_p(t, s) = \exp \left\{ \int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta\tau \right\} \quad \text{with} \quad \xi_h(z) = \begin{cases} \frac{\text{Log}(1+hz)}{h} & \text{if } h \neq 0 \\ z & \text{if } h = 0 \end{cases}. \tag{5.6}$$

In (5.6) ξ_h is the cylinder transformation defined for $h > 0$ from the set $\mathbb{C}_h := \{z \in \mathbb{C} : z \neq -\frac{1}{h}\}$ to the set $\mathbb{Z}_h := \{z \in \mathbb{C} : -\frac{\pi}{h} < \text{Im}(z) \leq \frac{\pi}{h}\}$. The set \mathbb{C}_h is called the Hilger complex plane [41].

Remark 106. For $h = 0$, $\mathbb{C}_0 := \mathbb{C}$.

Definition 107. [41, p. 52] Let $h > 0$ and $z \in \mathbb{C}_h$. The Hilger real part of z is defined by

$$\text{Re}_h(z) := \frac{|zh + 1| - 1}{h} \tag{5.7}$$

and the Hilger imaginary part of z by

$$\text{Im}_h(z) := \frac{\text{Arg}(zh + 1)}{h}, \tag{5.8}$$

where $\text{Arg}(z)$ denotes, as usual, the principal argument of z .

Remark 108. If we consider a time scale with graininess function $\mu(t) > 0$ and a z belonging to $\mathbb{C}_\mu := \left\{ z \in \mathbb{C} : z \neq -\frac{1}{\mu(t)} \right\}$, we can replace h by $\mu(t)$ in the right-hand sides of (5.7) and (5.8) and represent them, respectively, by $Re_\mu(z)$ and $Im_\mu(z)$.

Now we recall some properties of the exponential function stated in Theorem 2.36 of [41].

Theorem 109. [41] *If $p \in \mathcal{R}$, then,*

1. $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$;
2. $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
3. $\frac{1}{e_p(t, s)} = e_{\ominus p}(t, s)$.

Remark 110. Item 2 of Theorem 109 has special importance because it provides the ability to rewrite $e_p(\sigma(t), s)$ without the symbol $\sigma(t)$ and on that way simplify expressions of exponential function of elementary functions and by consequence the Laplace transform.

With all previous necessary concepts already given we are in conditions to understand the Laplace transform on time scales and provide some of their properties which are needed across this chapter.

The main results presented in this section were taken from [42] and/or [47].

Definition 111. [42] We define the *generalized Laplace transform* of a regulated function $f : \mathbb{T} \rightarrow \mathbb{C}$, where \mathbb{T} denotes a time scale which is unbounded above and contains zero, by

$$\mathcal{L}_{\mathbb{T}}[f](z) = F(z) := \int_0^\infty f(t)e_{\ominus z}(\sigma(t), 0)\Delta t \quad \text{for } z \in D[f], \quad (5.9)$$

where $D[f]$ consists of all $z \in \mathbb{C}$ for which the improper integral exists and for which $1 + \mu(t)z \neq 0$ for all $t \in \mathbb{T}$.

Remark 112. In view of Definition 111, the Laplace transform \mathcal{L} of Proposition 101 can be written as $\mathcal{L}_{\mathbb{R}}$.

Throughout the rest of chapter, \mathbb{T} is an arbitrary time scale with bounded graininess, i.e., $0 < \mu_{\min} \leq \mu(t) \leq \mu_{\max}$ for all $t \in \mathbb{T}$. Let $t_0 \in \mathbb{T}$ be fixed.

Definition 113. [47] The function $f : \mathbb{T} \rightarrow \mathbb{R}$ is said to be of *exponential type I* if there exist constants $M, c > 0$ such that $|f(t)| \leq Me^{ct}$. Furthermore, f is said to be of *exponential type II* if there exist constants $M, c > 0$ such that $|f(t)| \leq Me_c(t, 0)$.

The time scale exponential function $e_c(t, 0)$ is of type II while generalized polynomials $h_k(t, 0)$ are of type I.

Theorem 114. [47] *If f is of exponential type II with exponential constant c , then the delta integral $\int_0^\infty f(t)e_{\ominus z}(\sigma(t), 0)\Delta t$ converges absolutely for $z \in D$.*

Similarly to the classical Laplace transform (cf. formula (5.3)) we can state the following theorem:

Theorem 115. [42, Theorem 1.2] *If $f : \mathbb{T} \rightarrow \mathbb{C}$ is such that f^Δ is regulated, then*

$$\mathcal{L}_{\mathbb{T}}[f^\Delta](z) = z\mathcal{L}_{\mathbb{T}}[f](z) - f(0) \quad (5.10)$$

for all $z \in \mathcal{D}[f]$ such that $\lim_{t \rightarrow \infty} (f(t)e_{\ominus z}(t)) = 0$.

Remark 116. To prove Theorem 115 we first make use of the Laplace transform, next apply integration by parts formula, and then use some of the properties of the exponential function mentioned before to rewrite the result and get what the theorem claims.

Using the last result and mathematical induction one obtains the following result:

Proposition 117. *Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be such that f^{Δ^n} is regulated and F represents the generalized Laplace transform of f . Then,*

$$\mathcal{L}_{\mathbb{T}}[f^{\Delta^n}](z) = z^n F(z) - \sum_{k=0}^{n-1} z^{n-k-1} f^{\Delta^k}(0) \quad (5.11)$$

for all $z \in \mathcal{D}[f]$ such that $\lim_{t \rightarrow \infty} (f^{\Delta^k}(t)e_{\ominus z}(t)) = 0$ ($0 \leq k < n$).

Proof. The proof will be done by mathematical induction on n . Before starting the induction process it's important to note that the fact that f^{Δ^n} is regulated implies that all delta derivatives with order below n are also regulated. We know from Theorem 115 that

$$\mathcal{L}_{\mathbb{T}}[f^\Delta](z) = z\mathcal{L}_{\mathbb{T}}[f](z) - f(0).$$

Assuming that (5.11) is true we have

$$\begin{aligned}
 \mathcal{L}_{\mathbb{T}}[f^{\Delta^{n+1}}](z) &= z\mathcal{L}_{\mathbb{T}}[f^{\Delta^n}](z) - f^n(0) \\
 &= z \left(z^n F(z) - \sum_{k=0}^{n-1} z^{n-k-1} f^{\Delta^k}(0) \right) - f^{\Delta^n}(0) \\
 &= z^{n+1} F(z) - \sum_{k=0}^{n-1} z^{n-k} f^{\Delta^k}(0) - z^0 f^{\Delta^n}(0) \\
 &= z^{n+1} F(z) - \sum_{k=0}^n z^{n-k} f^{\Delta^k}(0).
 \end{aligned}$$

□

Using Theorem 118, Davis *et al.* [47] established Theorem 119 below.

Theorem 118. [47, Theorem 1.2] *Let F denote the generalized transform for $f : \mathbb{T} \rightarrow \mathbb{R}$. Then:*

1. $F(z)$ is analytic in $Re_{\mu}(z) > Re_{\mu}(c)$;
2. $F(z)$ is bounded in $Re_{\mu}(z) > Re_{\mu}(c)$;
3. $\lim_{|z| \rightarrow \infty} F(z) = 0$.

Since our idea is to apply the Laplace transform theory on time scales, particularly the inverse of Laplace transform on time scales, to define the fractional integral (cf. Definition 125) and fractional derivative (cf. Definition 126) on time scales, Theorem 119 plays an important role in our work by providing sufficient conditions for the existence of such inverse and a formula to evaluate it.

Theorem 119 (Inversion formula of the Laplace transform [47]). *Suppose that F is analytic in the region $Re_{\mu}(z) > Re_{\mu}(c)$ and $F(z) \rightarrow 0$ uniformly as $|z| \rightarrow \infty$ in this region. Assume F has finitely many regressive poles of finite order $\{z_1, z_2, \dots, z_n\}$ and $\tilde{F}_{\mathbb{R}}(z)$ is the transform of the function $\tilde{f}(t)$ on \mathbb{R} that corresponds to the transform $F(z) = F_{\mathbb{T}}(z)$ of $f(t)$ on \mathbb{T} . If*

$$\int_{c-i\infty}^{c+i\infty} \left| \tilde{F}_{\mathbb{R}}(z) \right| |dz| < \infty,$$

then

$$f(t) = \sum_{i=1}^n Res_{z=z_i} e_z(t, 0) F(z)$$

has transform $F(z)$ for all z with $Re(z) > c$.

Example 120. [47, Example 1.1] Suppose $F(z) = \frac{1}{z^2}$. Then

$$\mathcal{L}_{\mathbb{T}}^{-1}[F](z) = f(t) = \text{Res}_{z=0} \frac{e_z(t, 0)}{z^2} = t.$$

Applying Theorem 119 for $F(z) = \frac{1}{z^3}$ we get

$$\mathcal{L}_{\mathbb{T}}^{-1}[F](z) = f(t) = h_2(t, 0).$$

Remark 121. Using induction arguments over the results of Example 120 it's easy to prove that the inverse transform of $F(z) = \frac{1}{z^{k+1}}$, $k \in \mathbb{N}$, is $h_k(\cdot, 0)$.

Remark 122. The inversion formula also gives the claimed inverses for any of the elementary functions that were presented in the table of Laplace transform in [42].

The following lemma establish a relation between the solution of (5.5) with power series on time scales.

Lemma 123. [39, Lemma 4.4] For all $z \in \mathbb{C}$ and $t \in \mathbb{T}$ with $t \geq \alpha$, the initial value problem

$$y^\Delta = zy, \quad y(\alpha) = 1 \tag{5.12}$$

has a unique solution y that is represented in the form

$$y(t) = \sum_{k=0}^{+\infty} z^k h_k(t, \alpha) \tag{5.13}$$

and satisfies the inequality

$$|y(t)| \leq e^{|z|(t-\alpha)}. \tag{5.14}$$

Remark 124. Let $\alpha = 0$. If we use jointly Lemma 123 and the well-known fact that the unique solution of IVP (5.5) is $e_z(t, 0)$ we have the equality

$$e_z(t, 0) = \sum_{k=0}^{+\infty} z^k h_k(t, 0)$$

that allows us to see that generalized polynomials $h_k(t, 0)$ is the reason to have different inverse Laplace images when we have different time scales.

5.3 Main Results

We begin by introducing the definition of fractional integral and fractional derivative on an arbitrary time scale \mathbb{T} .

5.3.1 Fractional derivative and integral on time scales

Simultaneously, the generalized Laplace transform on time scales gives unification and extension of the classical results. Important to us, the Laplace transform of the Δ -derivative is given by the formula $\mathcal{L}_{\mathbb{T}}[f^{\Delta}](z) = zF(z) - f(0)$. Our idea is to define the fractional derivative on time scales via the inverse Laplace transform formula for the complex function $G(z) = z^{\alpha}\mathcal{L}_{\mathbb{T}}[f](z) - f(0^+)z^{\alpha-1}$. Furthermore, for $\alpha \in (n-1, n]$, $n \in \mathbb{N}$, we use a generalization of (5.11) to define fractional derivatives on times scales for higher orders α .

Definition 125 (Fractional integral on time scales). Let $\alpha > 0$, \mathbb{T} be a time scale, and $f : \mathbb{T} \rightarrow \mathbb{R}$. The fractional integral of f of order α on the time scale \mathbb{T} , denoted by $I_{\mathbb{T}}^{\alpha}f$, is defined by

$$I_{\mathbb{T}}^{\alpha}f(t) = \mathcal{L}_{\mathbb{T}}^{-1} \left[\frac{F(z)}{z^{\alpha}} \right] (t).$$

Definition 126 (Fractional derivative on time scales). Let \mathbb{T} be a time scale, $F(z) = \mathcal{L}_{\mathbb{T}}[f](z)$, and $\alpha \in (n-1, n]$, $n \in \mathbb{N}$. The fractional derivative of function f of order α on the time scale \mathbb{T} , denoted by $f^{(\alpha)}$, is defined by

$$f^{(\alpha)}(t) = \mathcal{L}_{\mathbb{T}}^{-1} \left[z^{\alpha}F(z) - \sum_{k=0}^{n-1} f^{\Delta^k}(0^+)z^{\alpha-k-1} \right] (t). \quad (5.15)$$

Remark 127. For $\alpha \in (0, 1]$ we have

$$f^{(\alpha)}(t) = \mathcal{L}_{\mathbb{T}}^{-1} [z^{\alpha}F(z) - f(0^+)z^{\alpha-1}] (t).$$

Moreover, if we use $\mathbb{T} = \mathbb{R}$ we have

$$f^{(\alpha)}(t) = \mathcal{L}_{\mathbb{R}}^{-1} [z^{\alpha}F(z) - f(0^+)z^{\alpha-1}] (t) = \mathcal{L}_{\mathbb{R}}^{-1} [\mathcal{L}_{[0^+}^C D_x^{\alpha} f](z)] (t) = ({}_0^C D_x^{\alpha} f)(t).$$

5.3.2 Properties

We begin with two trivial but important remarks about the fractional integral and the fractional derivative operators just introduced.

Remark 128. As the inverse Laplace transform is linear, we also have linearity for the new fractional integral and derivative:

$$\begin{aligned} I_{\mathbb{T}}^{\alpha}(af + bg)(t) &= aI_{\mathbb{T}}^{\alpha}f(t) + bI_{\mathbb{T}}^{\alpha}g(t), \\ (af + bg)^{(\alpha)}(t) &= af^{(\alpha)}(t) + bg^{(\alpha)}(t). \end{aligned}$$

Remark 129. Since $h_0(t) \equiv 1$ for any time scale \mathbb{T} , from the definition of Laplace transform and the fractional derivative we conclude that $h_0^{(\alpha)}(t) = 0$. For $\alpha > 1$ one has $h_1^{(\alpha)}(t) = 0$.

We now prove several important properties of the fractional integrals and fractional derivatives on arbitrary time scales.

Proposition 130. *Let $\alpha \in (n - 1, n]$, $n \in \mathbb{N}$. If $k \leq n - 1$, then*

$$h_k^{(\alpha)}(t, 0) = 0.$$

Proof. From (5.15) it follows that

$$\begin{aligned} h_k^{(\alpha)}(t, 0) &= \mathcal{L}_{\mathbb{T}}^{-1} \left[\frac{z^{\alpha}}{z^{k+1}} - \sum_{i=0}^{n-1} h_k^{\Delta^i}(0) z^{\alpha-i-1} \right] (t) = \mathcal{L}_{\mathbb{T}}^{-1} \left[\frac{z^{\alpha}}{z^{k+1}} - \sum_{i=0}^k h_{k-i}(0) z^{\alpha-i-1} \right] (t) \\ &= \mathcal{L}_{\mathbb{T}}^{-1} \left[\frac{z^{\alpha}}{z^{k+1}} - z^{\alpha-k-1} \right] (t) = 0. \end{aligned}$$

□

Proposition 131. *Let $\alpha \in (n - 1, n]$, $n \in \mathbb{N}$. If $k \geq n$, then*

$$h_k^{(\alpha)}(t, 0) = \mathcal{L}_{\mathbb{T}}^{-1} \left[\frac{1}{z^{k+1-\alpha}} \right] (t).$$

Proof. From (5.15) we have

$$\begin{aligned} h_k^{(\alpha)}(t) &= \mathcal{L}_{\mathbb{T}}^{-1} \left[\frac{z^{\alpha}}{z^{k+1}} - \sum_{i=0}^{n-1} h_k^{\Delta^i}(0) z^{\alpha-i-1} \right] (t) = \mathcal{L}_{\mathbb{T}}^{-1} \left[\frac{z^{\alpha}}{z^{k+1}} - \sum_{i=0}^{n-1} h_{k-i}(0) z^{\alpha-i-1} \right] (t) \\ &= \mathcal{L}_{\mathbb{T}}^{-1} \left[\frac{z^{\alpha}}{z^{k+1}} \right] (t) = \mathcal{L}_{\mathbb{T}}^{-1} \left[\frac{1}{z^{k+1-\alpha}} \right] (t). \end{aligned}$$

□

Proposition 132. *Let $\alpha \in (n - 1, n]$, $n \in \mathbb{N}$. If $c(t) \equiv m$, $m \in \mathbb{R}$, then*

$$c^{(\alpha)}(t) = 0.$$

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Proof. From the linearity of the inverse Laplace transform and the fact that

$$h_0^{(\alpha)}(t, 0) = 0$$

it follows that

$$c^{(\alpha)}(t) = (m \cdot 1)^{(\alpha)} = (m \cdot h_0(t, 0))^{(\alpha)} = m \cdot h_0^{(\alpha)}(t, 0) = m \cdot 0 = 0.$$

□

Proposition 133. *Let $\alpha, \beta > 0$. Then,*

$$I_{\mathbb{T}}^{\beta} (I_{\mathbb{T}}^{\alpha} f) (t) = I_{\mathbb{T}}^{\alpha+\beta} f(t).$$

Proof. By definition we have

$$I_{\mathbb{T}}^{\beta} (I_{\mathbb{T}}^{\alpha} f) (t) = \mathcal{L}_{\mathbb{T}}^{-1} [z^{-\beta} \mathcal{L}_{\mathbb{T}} [I_{\mathbb{T}}^{\alpha} f]] (t) = \mathcal{L}_{\mathbb{T}}^{-1} [s^{-\alpha-\beta} F(z)] (t) = I_{\mathbb{T}}^{\alpha+\beta} f(t).$$

□

Proposition 134. *Let $\alpha, \beta \in (0, 1]$.*

(a) *If $\alpha + \beta \leq 1$, then*

$$(f^{(\alpha)})^{(\beta)} (t) = f^{(\alpha+\beta)}(t) - \mathcal{L}_{\mathbb{T}}^{-1} [z^{\beta-1} f^{(\alpha)}(0)] (t).$$

(b) *If $1 < \alpha + \beta \leq 2$, then*

$$(f^{(\alpha)})^{(\beta)} (t) = f^{(\alpha+\beta)}(t) - \mathcal{L}_{\mathbb{T}}^{-1} [z^{\beta-1} f^{(\alpha)}(0)] (t) + \mathcal{L}_{\mathbb{T}}^{-1} [z^{\alpha+\beta-2} f^{\Delta}(0)] (t).$$

(c) *If $\beta \in (0, 1]$, then*

$$(f^{\Delta})^{(\beta)} (t) = f^{(\beta+1)}(t).$$

Proof. (a) Let $\alpha + \beta \leq 1$. Then

$$\begin{aligned} (f^{(\alpha)})^{(\beta)} (t) &= \mathcal{L}_{\mathbb{T}}^{-1} [z^{\beta} \mathcal{L} [f^{(\alpha)}] (z) - z^{\beta-1} f^{(\alpha)}(0)] (t) \\ &= \mathcal{L}_{\mathbb{T}}^{-1} [z^{\alpha+\beta} F(z) - z^{\alpha+\beta-1} f(0)] (t) - \mathcal{L}_{\mathbb{T}}^{-1} [z^{\beta-1} f^{(\alpha)}(0)] (t) \\ &= f^{(\alpha+\beta)}(t) - \mathcal{L}_{\mathbb{T}}^{-1} [z^{\beta-1} f^{(\alpha)}(0)] (t). \end{aligned}$$

(b) For $1 < \alpha + \beta \leq 2$ we have

$$f^{(\alpha+\beta)}(t) = \mathcal{L}_{\mathbb{T}}^{-1} [z^{\alpha+\beta} \mathcal{L}[f](z) - z^{\alpha+\beta-1} f(0) - z^{\alpha+\beta-2} f^{\Delta}(0)](t).$$

Hence,

$$(f^{(\alpha)})^{(\beta)}(t) = f^{(\alpha+\beta)}(t) + \mathcal{L}_{\mathbb{T}}^{-1} [z^{\alpha+\beta-2} f^{\Delta}(0)](t) - \mathcal{L}_{\mathbb{T}}^{-1} [z^{\beta-1} f^{(\alpha)}(0)](t).$$

(c) The intended relation follows from (b) by choosing $\alpha = 1$. \square

In general $(f^{(\alpha)})^{(\beta)}(t) \neq (f^{(\beta)})^{(\alpha)}(t)$. However, the equality holds in particular cases:

Proposition 135. *If $\alpha + \beta \leq 1$ and $f(0) = 0$, then $(f^{(\alpha)})^{(\beta)}(t) = (f^{(\beta)})^{(\alpha)}(t)$.*

Proof. It follows from item (a) of Proposition 134. \square

Proposition 136. *Let $\alpha \in (n-1, n]$, $n \in \mathbb{N}$, and $\lim_{t \rightarrow 0^+} f^{\Delta^k}(t) = f^{\Delta^k}(0^+)$ exist, $k = 0, \dots, n-1$. The following equality holds:*

$$I_{\mathbb{T}}^{\alpha}(f^{(\alpha)})(t) = f(t) - \sum_{k=0}^{n-1} f^{\Delta^k}(0^+) h_k(t).$$

Proof. Let $F(z) = \mathcal{L}_{\mathbb{T}}[f](z)$. Then,

$$\begin{aligned} I_{\mathbb{T}}^{\alpha}(f^{(\alpha)})(t) &= \mathcal{L}_{\mathbb{T}}^{-1} [z^{-\alpha} \mathcal{L}_{\mathbb{T}}[f^{(\alpha)}](z)](t) = \mathcal{L}_{\mathbb{T}}^{-1} \left[F(z) - \sum_{k=0}^{n-1} f^{\Delta^k}(0^+) \frac{z^{\alpha-k-1}}{z^{\alpha}} \right](t) \\ &= f(t) - \sum_{k=0}^{n-1} f^{\Delta^k}(0^+) \mathcal{L}_{\mathbb{T}}^{-1} \left[\frac{1}{z^{k+1}} \right](t) = f(t) - \sum_{k=0}^{n-1} f^{\Delta^k}(0^+) h_k(t). \end{aligned}$$

\square

Proposition 137. *Let $\alpha \in (n-1, n]$, $n \in \mathbb{N}$, and $F(z) = \mathcal{L}_{\mathbb{T}}[f](z)$. If $\lim_{z \rightarrow \infty} F(z) = 0$ and $\lim_{z \rightarrow \infty} \frac{F(z)}{z^{\alpha-k}} = 0$, $k \in \{1, \dots, n\}$, then $(I_{\mathbb{T}}^{\alpha} f)^{(\alpha)}(t) = f(t)$.*

Proof. Firstly let us notice that $\lim_{t \rightarrow 0^+} (I_{\mathbb{T}}^{\alpha} f)^{\Delta^k}(t) = 0$ for $k \in \{0, \dots, n-1\}$. For that we check that $\lim_{z \rightarrow \infty} z \mathcal{L}[I_{\mathbb{T}}^{\alpha} f](z) = \lim_{z \rightarrow \infty} z \frac{F(z)}{z^{\alpha}} = 0$. Nextly let us assume that it holds for $i = 0, \dots, k-1$ and let us calculate

$$\begin{aligned} \lim_{z \rightarrow \infty} z \mathcal{L} \left[(I_{\mathbb{T}}^{\alpha} f)^{\Delta^k} \right](z) &= \lim_{z \rightarrow \infty} z \left(z^k \frac{F(z)}{z^{\alpha}} - \sum_{i=0}^{k-1} z^{k-1-i} (I_{\mathbb{T}}^{\alpha})^{\Delta^i}(0) \right) \\ &= \lim_{z \rightarrow \infty} z \left(z^k \frac{F(z)}{z^{\alpha}} \right) = \lim_{z \rightarrow \infty} \frac{F(z)}{z^{\alpha-k-1}} = 0. \end{aligned}$$

Then we easily conclude that

$$(I_{\mathbb{T}}^{\alpha} f)^{(\alpha)}(t) = \mathcal{L}_{\mathbb{T}}^{-1} \left[z^{\alpha} \frac{F(z)}{z^{\alpha}} \right] (t) = f(t).$$

□

The convolution of two functions $f : \mathbb{T} \rightarrow \mathbb{R}$ and $g : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}$ on time scales, where g is rd-continuous with respect to the first variable, is defined in [40, 47]:

$$(f * g)(t) = \int_0^t f(\tau)g(t, \sigma(\tau))\Delta\tau.$$

As function g we can consider, e.g., $e_c(t, t_0)$ or $h_k(t, t_0)$.

Proposition 138. *Let $t_0 \in \mathbb{T}$. If $\alpha \in (0, 1)$, then*

$$(f * g(\cdot, t_0))^{(\alpha)}(t) = (f^{(\frac{\alpha}{2})} * g^{(\frac{\alpha}{2})}(\cdot, t_0))(t) = (f^{(\alpha)} * g(\cdot, t_0))(t),$$

where we assume the existence of the involved derivatives.

Proof. From the convolution theorem for the generalized Laplace transform [47, Theorem 2.1],

$$\mathcal{L}_{\mathbb{T}} \left[(f * g(\cdot, t_0))^{(\alpha)} \right] (z) = z^{\alpha} F(z)G(z).$$

Hence,

$$\begin{aligned} (f * g(\cdot, t_0))^{(\alpha)}(t) &= \mathcal{L}_{\mathbb{T}}^{-1} [z^{\alpha} F(z)G(z)](t) = \mathcal{L}_{\mathbb{T}}^{-1} [z^{\frac{\alpha}{2}} F(z)](t) \mathcal{L}_{\mathbb{T}}^{-1} [z^{\frac{\alpha}{2}} G(z)](t) \\ &= (f^{(\alpha/2)} * g^{(\alpha/2)}(\cdot, t_0))(t). \end{aligned}$$

Equivalently, we can write

$$\mathcal{L}_{\mathbb{T}}^{-1} [z^{\alpha} F(z)G(z)](t) = \mathcal{L}_{\mathbb{T}}^{-1} [z^{\alpha} F(z)](t) \mathcal{L}_{\mathbb{T}}^{-1} [G(z)](t) = (f^{(\alpha)} * g(\cdot, t_0))(t).$$

□

5.4 State of the Art

The results of this chapter are already published in [35] and were presented by the author in a seminar with the title *Fractional Derivatives and Integrals on Time Scales via the Inverse Generalized Laplace Transform* on June 9, 2011, at the Faculty of Computer Science of Białystok University of Technology, Białystok, Poland.

Chapter 6

Conclusions and future work

In this thesis we define fractional sum and difference operators for the time scales $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = (h\mathbb{Z})_a$ and prove some properties for them. We also established and proved first and second order necessary optimality conditions for problems of the calculus of variations on both time scales. They show that the solutions of the fractional problems coincide with the solutions of the corresponding non-fractional variational problems when the order of the discrete derivatives is an integer value. We believe that the present work will potentiate further research in the fractional calculus of variations where much remains to be done. With respect to the fractional derivatives and integrals on an arbitrary time scale \mathbb{T} , we introduce a fractional calculus on time scales using the theory of delta dynamic equations and the inverse Laplace transform on time scales. The basic notions of fractional order integral and fractional order derivative on an arbitrary time scale are given, and some properties of the new introduced fractional operators are proved.

Some possible directions of research are:

- to extend the results on fractional variational problems for higher-order problems of the calculus of variations [27, 44, 52, 84] with fractional discrete derivatives of any order;
- to explore new possibilities of research for fractional continuous variational problems, in particular, to get a fractional continuous Legendre necessary optimality condition (an interesting open question);
- to explore several areas of application, for example, in signal processing, where fractional derivatives of a discrete-time signal are particularly useful to describe noise

processes [99];

- to find a general formulation leading to the specification of ARMA parameters (as pointed out by one of the referees of [33]);
- to study optimality conditions for more general variable endpoint variational problems [65, 79, 80] and isoperimetric problems [12, 13];
- to obtain fractional sufficient optimality conditions of Jacobi type and a version of Noether's theorem [32, 46, 55, 57] for discrete-time fractional variational problems;
- to explore direct methods of optimization for absolute extrema [38, 81, 113];
- to generalize our first and second order optimality conditions to a fractional Lagrangian possessing delay terms [1, 29];
- to extend our results of discrete fractional variational problems to problems involving Caputo difference operators [17] or even Riemann-Liouville and Caputo together on the same Lagrangian.

Another line of research, where we just have few things done [36], is related with the new discrete fractional diamond sum (cf. Definition 142 below) in the following topics:

- to investigate the discrete fractional diamond difference introduced in [36] and generalize the new discrete diamond fractional operator to an arbitrary time scale \mathbb{T} ;
- to investigate the usefulness of modeling with fractional equations [101] and study corresponding fractional diamond variational principles;
- to extend the results presented in Chapters 3 and 4 to an arbitrary time scale;
- to study and develop results on other subjects of mathematics using the theory of discrete fractional calculus.

All the results in Chapters 3, 4 and 5 were obtained for delta-calculus but could easily be rewritten using the operator nabla. However, in our opinion, doesn't make sense to have results for Δ and for ∇ separately. Motivated by the diamond-alpha dynamic derivative on time scales [16, 77, 88] and the fractional derivative of [78, 83], we introduce a new operator that unify and extend the operators Δ and ∇ by using a convex linear combination of

them. To exemplify what we propose, we introduce a more general fractional sum operator (for $\mathbb{T} = \mathbb{Z}$), making use of the symbol, ${}_{\gamma}\diamond$ (cf. Definition 142). With this new operator it's not our intention to change what was done in Δ and ∇ cases – as particular cases, results on delta and nabla discrete fractional calculus are obtained from the new operator – but just to unify the two cases in one.

Looking to the literature of discrete fractional difference operators, two approaches are found (see, e.g., [22, 33]): one using the Δ point of view (sometimes called the forward fractional difference approach), another using the ∇ perspective (sometimes called the backward fractional difference approach). When $\gamma = 1$ our ${}_{\gamma}\diamond$ operator is reduced to the Δ one; when $\gamma = 0$ the ${}_{\gamma}\diamond$ operator coincides with the corresponding ∇ fractional sum.

Analogously to Definition 2, one considers the discrete nabla fractional sum operator:

Definition 139. [22] The discrete nabla fractional sum operator is defined by

$$(\nabla_a^{-\beta} f)(t) = \frac{1}{\Gamma(\beta)} \sum_{s=a}^t (t - \rho(s))^{\overline{\beta-1}} f(s),$$

where $\beta > 0$. Here f is defined for $s = a \pmod{1}$ and $\nabla_a^{-\beta} f$ is defined for $t = a \pmod{1}$.

Remark 140. Let $\mathbb{N}_a = \{a, a+1, a+2, \dots\}$. The operator $\nabla_a^{-\beta}$ maps functions defined on \mathbb{N}_a to functions defined on \mathbb{N}_a . The fact that f and $\nabla_a^{-\beta} f$ have the same domain, while f and $\Delta_a^{-\alpha} f$ do not, explains why some authors prefer the nabla approach.

The next result gives a relation between the delta fractional sum and the nabla fractional sum operators.

Lemma 141. [22] Let $0 \leq m-1 < \nu \leq m$, where m denotes an integer. Let a be a positive integer, and $y(t)$ be defined on $t \in \mathbb{N}_a = \{a, a+1, a+2, \dots\}$. The following statement holds: $(\Delta_a^{-\nu} y)(t+\nu) = (\nabla_a^{-\nu} y)(t)$, $t \in \mathbb{N}_a$.

With the help of this lemma we are able to rewrite our results of Chapter 3 to the ∇ operator. Our intention is not to have results for Δ and ∇ separately but, instead, to extend and unify them through the following operator:

Definition 142. The diamond- γ fractional operator of order (α, β) is given, when applied to a function f at point t , by

$$({}_{\gamma}\diamond_a^{-\alpha, -\beta} f)(t) = \gamma (\Delta_a^{-\alpha} f)(t+\alpha) + (1-\gamma) (\nabla_a^{-\beta} f)(t),$$

where $\alpha > 0$, $\beta > 0$, and $\gamma \in [0, 1]$. Here, both f and ${}_{\gamma}\diamond_a^{-\alpha, -\beta} f$ are defined for $t = a \pmod{1}$.

Remark 143. Similarly to the nabla fractional operator, our operator ${}_{\gamma}\diamond_a^{-\alpha, -\beta}$ maps functions defined on \mathbb{N}_a to functions defined on \mathbb{N}_a , $\mathbb{N}_a = \{a, a + 1, a + 2, \dots\}$ for a a given real number.

Remark 144. The new diamond fractional operator of Definition 142 gives, as particular cases, the operator of Definition 2 for $\gamma = 1$,

$$({}_1\diamond_a^{-\alpha, -\beta} f)(t) = (\Delta_a^{-\alpha} f)(t + \alpha), \quad t \equiv a \pmod{1},$$

and the operator of Definition 139 for $\gamma = 0$,

$$({}_0\diamond_a^{-\alpha, -\beta} f)(t) = (\nabla_a^{-\beta} f)(t), \quad t \equiv a \pmod{1}.$$

The next theorems give important properties of the new, more general, discrete fractional operator ${}_{\gamma}\diamond_a^{-\alpha, -\beta}$.

Theorem 145. *Let f and g be real functions defined on \mathbb{N}_a , $\mathbb{N}_a = \{a, a + 1, a + 2, \dots\}$ for a a given real number. The following equality holds:*

$$({}_{\gamma}\diamond_a^{-\alpha, -\beta}(f + g))(t) = ({}_{\gamma}\diamond_a^{-\alpha, -\beta} f)(t) + ({}_{\gamma}\diamond_a^{-\alpha, -\beta} g)(t).$$

Proof. The intended equality follows from the definition of diamond- γ fractional sum of

order (α, β) :

$$\begin{aligned}
({}_\gamma \diamond_a^{-\alpha, -\beta} (f + g))(t) &= \gamma (\Delta_a^{-\alpha} (f + g))(t + \alpha) + (1 - \gamma) (\nabla_a^{-\beta} (f + g))(t) \\
&= \frac{\gamma}{\Gamma(\alpha)} \sum_{s=a}^t (t + \alpha - \sigma(s))^{\alpha-1} (f(s) + g(s)) \\
&\quad + \frac{1 - \gamma}{\Gamma(\beta)} \sum_{s=a}^t (t - \rho(s))^{\beta-1} (f(s) + g(s)) \\
&= \frac{\gamma}{\Gamma(\alpha)} \sum_{s=a}^t (t + \alpha - \sigma(s))^{\alpha-1} f(s) + \frac{\gamma}{\Gamma(\alpha)} \sum_{s=a}^t (t + \alpha - \sigma(s))^{\alpha-1} g(s) \\
&\quad + \frac{1 - \gamma}{\Gamma(\beta)} \sum_{s=a}^t (t - \rho(s))^{\beta-1} f(s) + \frac{1 - \gamma}{\Gamma(\beta)} \sum_{s=a}^t (t - \rho(s))^{\beta-1} g(s) \\
&= \left[\frac{\gamma}{\Gamma(\alpha)} \sum_{s=a}^t (t + \alpha - \sigma(s))^{\alpha-1} f(s) + \frac{1 - \gamma}{\Gamma(\beta)} \sum_{s=a}^t (t - \rho(s))^{\beta-1} f(s) \right] \\
&\quad + \left[\frac{\gamma}{\Gamma(\alpha)} \sum_{s=a}^t (t + \alpha - \sigma(s))^{\alpha-1} g(s) + \frac{1 - \gamma}{\Gamma(\beta)} \sum_{s=a}^t (t - \rho(s))^{\beta-1} g(s) \right] \\
&= ({}_\gamma \diamond_a^{-\alpha, -\beta} f)(t) + ({}_\gamma \diamond_a^{-\alpha, -\beta} g)(t).
\end{aligned}$$

□

Theorem 146. Let $f(t) = k$ on \mathbb{N}_a , k a constant. The following equality holds:

$$({}_\gamma \diamond_a^{-\alpha, -\beta} f)(t) = \gamma \frac{\Gamma(t - a + 1 + \alpha)k}{\Gamma(\alpha + 1)\Gamma(t - a + 1)} + (1 - \gamma) \frac{\Gamma(t - a + 1 + \beta)k}{\Gamma(\beta + 1)\Gamma(t - a + 1)}.$$

Proof. By definition of diamond- γ fractional sum of order (α, β) , we have

$$\begin{aligned}
({}_\gamma \diamond_a^{-\alpha, -\beta} k)(t) &= \gamma (\Delta_a^{-\alpha} k)(t + \alpha) + (1 - \gamma) (\nabla_a^{-\beta} k)(t) \\
&= \frac{\gamma}{\Gamma(\alpha)} \sum_{s=0}^t k(t + \alpha - \sigma(s))^{\alpha-1} + \frac{1 - \gamma}{\Gamma(\beta)} \sum_{s=0}^t k(t - \rho(s))^{\beta-1} \\
&= \gamma \frac{\Gamma(t - a + 1 + \alpha)}{\alpha \Gamma(\alpha) \Gamma(t - a + 1)} k + (1 - \gamma) \frac{\Gamma(t - a + 1 + \beta)}{\beta \Gamma(\beta) \Gamma(t - a + 1)} k \\
&= \gamma \frac{\Gamma(t - a + 1 + \alpha)}{\Gamma(\alpha + 1) \Gamma(t - a + 1)} k + (1 - \gamma) \frac{\Gamma(t - a + 1 + \beta)}{\Gamma(\beta + 1) \Gamma(t - a + 1)} k.
\end{aligned}$$

□

Corollary 147. Let $f(t) \equiv k$ for a certain constant k . Then,

$$(\Delta_a^{-\alpha} f)(t + \alpha) = \frac{\Gamma(t - a + 1 + \alpha)}{\Gamma(\alpha + 1) \Gamma(t - a + 1)} k. \tag{6.1}$$

Proof. The result follows from Theorem 146 choosing $\gamma = 1$ and recalling that

$$({}_1\Diamond_a^{-\alpha, -\beta}k)(t) = (\Delta_a^{-\alpha}k)(t + \alpha).$$

□

Remark 148. In the particular case when $a = 0$, equality (6.1) coincides with the result of [86, Section 5].

The fractional nabla result analogous to Corollary 147 is easily obtained:

Corollary 149. *If k is a constant, then*

$$(\nabla_a^{-\beta}k)(t) = \frac{\Gamma(t - a + 1 + \beta)}{\Gamma(\beta + 1)\Gamma(t - a + 1)}k.$$

Proof. The result follows from Theorem 146 choosing $\gamma = 0$ and recalling that

$$({}_0\Diamond_a^{-\alpha, -\beta}k)(t) = (\nabla_a^{-\beta}k)(t).$$

□

Theorem 150. *Let f be a real valued function and $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$. Then,*

$$\begin{aligned} (\gamma\Diamond_a^{-\alpha_1, -\beta_1}(\gamma\Diamond_a^{-\alpha_2, -\beta_2}f))(t) &= \gamma(\gamma\Diamond_a^{-(\alpha_1+\alpha_2), -(\beta_1+\beta_2)}f)(t) \\ &\quad + (1 - \gamma)(\gamma\Diamond_a^{-(\alpha_1+\beta_2), -(\beta_1+\beta_2)}f)(t). \end{aligned}$$

Proof. Direct calculations show the intended relation:

$$\begin{aligned} (\gamma\Diamond_a^{-\alpha_1, -\beta_1}(\gamma\Diamond_a^{-\alpha_2, -\beta_2}f))(t) &= \gamma(\Delta_a^{-\alpha_1}(\gamma\Diamond_a^{-\alpha_2, -\beta_2}f))(t + \alpha_1) \\ &\quad + (1 - \gamma)(\nabla_a^{-\beta_1}(\gamma\Diamond_a^{-\alpha_2, -\beta_2}f))(t) \\ &= \gamma^2(\Delta_a^{-\alpha_1}(\Delta_a^{-\alpha_2}f))(t + \alpha_1 + \alpha_2) + \gamma(1 - \gamma)(\Delta_a^{-\alpha_1}(\nabla_a^{-\beta_2}f))(t + \alpha_1) \\ &\quad + (1 - \gamma)\gamma(\nabla_a^{-\beta_1}(\Delta_a^{-\alpha_2}f))(t + \alpha_2) + (1 - \gamma)^2(\nabla_a^{-\beta_1}(\nabla_a^{-\beta_2}f))(t) \\ &= \gamma^2(\Delta_a^{-(\alpha_1+\alpha_2)}f)(t + \alpha_1 + \alpha_2) + \gamma(1 - \gamma)(\Delta_a^{-\alpha_1}(\Delta_a^{-\beta_2}f))(t + \alpha_1 + \beta_2) \\ &\quad + (1 - \gamma)\gamma(\nabla_a^{-\beta_1}(\nabla_a^{-\alpha_2}f))(t) + (1 - \gamma)^2(\nabla_a^{-(\beta_1+\beta_2)}f)(t) \\ &= \gamma^2(\Delta_a^{-(\alpha_1+\alpha_2)}f)(t + \alpha_1 + \alpha_2) + \gamma(1 - \gamma)(\Delta_a^{-(\alpha_1+\beta_2)}f)(t + \alpha_1 + \beta_2) \\ &\quad + (1 - \gamma)\gamma(\nabla_a^{-(\beta_1+\alpha_2)}f)(t) + (1 - \gamma)^2(\nabla_a^{-(\beta_1+\beta_2)}f)(t) \\ &= \gamma[\gamma(\Delta_a^{-(\alpha_1+\alpha_2)}f)(t + \alpha_1 + \alpha_2) + (1 - \gamma)(\nabla_a^{-(\beta_1+\alpha_2)}f)(t)] \\ &\quad + (1 - \gamma)[\gamma(\Delta_a^{-(\alpha_1+\beta_2)}f)(t + \alpha_1 + \beta_2) + (1 - \gamma)(\nabla_a^{-(\beta_1+\beta_2)}f)(t)]. \end{aligned}$$

□

Remark 151. If $\gamma = 0$, then $(\gamma \diamond_a^{-\alpha_1, -\beta_1} (\gamma \diamond_a^{-\alpha_2, -\beta_2} f))(t) = (\nabla_a^{-(\beta_1 + \beta_2)} f)(t)$.

Remark 152. If $\gamma = 1$, then $(\gamma \diamond_a^{-\alpha_1, -\beta_1} (\gamma \diamond_a^{-\alpha_2, -\beta_2} f))(t) = (\Delta_a^{-(\alpha_1 + \alpha_2)} f)(t + \alpha_1 + \alpha_2)$.

Remark 153. If $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$, then

$$(\gamma \diamond_a^{-\alpha, -\beta} (\gamma \diamond_a^{-\alpha, -\beta} f))(t) = (\gamma \diamond_a^{-\alpha, -\beta} f)(t).$$

We now prove a general Leibniz formula.

Theorem 154 (Leibniz formula). *Let f and g be real valued functions, $0 < \alpha, \beta < 1$. For all t such that $t = a \pmod{1}$, the following equality holds:*

$$\begin{aligned} (\gamma \diamond_a^{-\alpha, -\beta} (fg))(t) &= \gamma \sum_{k=0}^{\infty} \binom{-\alpha}{k} [(\nabla^k g)(t)] \cdot [(\Delta_a^{-(\alpha+k)} f)(t + \alpha + k)] \\ &\quad + (1 - \gamma) \sum_{k=0}^{\infty} \binom{-\beta}{k} [(\nabla^k g)(t)] [(\Delta_a^{-(\beta+k)} f)(t + \beta + k)], \end{aligned} \quad (6.2)$$

where

$$\binom{u}{v} = \frac{\Gamma(u+1)}{\Gamma(v+1)\Gamma(u-v+1)}.$$

Proof. By definition of the diamond fractional sum,

$$\begin{aligned} (\gamma \diamond_a^{-\alpha, -\beta} (fg))(t) &= \gamma (\Delta_a^{-\alpha} (fg))(t + \alpha) + (1 - \gamma) (\nabla_a^{-\beta} (fg))(t) \\ &= \frac{\gamma}{\Gamma(\alpha)} \sum_{s=a}^t (t + \alpha - \sigma(s))^{\alpha-1} f(s)g(s) \\ &\quad + \frac{1 - \gamma}{\Gamma(\beta)} \sum_{s=a}^t (t - \rho(s))^{\beta-1} f(s)g(s). \end{aligned}$$

By Taylor's expansion of $g(s)$ [19],

$$g(s) = \sum_{k=0}^{\infty} \frac{(s-t)^{\bar{k}}}{k!} (\nabla^k g)(t) = \sum_{k=0}^{\infty} (-1)^k \frac{(t-s)^{(k)}}{k!} (\nabla^k g)(t).$$

Substituting the Taylor series of $g(s)$ at t ,

$$\begin{aligned} (\gamma \diamond_a^{-\alpha, -\beta} (fg))(t) &= \frac{\gamma}{\Gamma(\alpha)} \sum_{s=a}^t (t + \alpha - \sigma(s))^{\alpha-1} f(s) \left[\sum_{k=0}^{\infty} (-1)^k (t-s)^{(k)} \frac{(\nabla^k g)(t)}{k!} \right] \\ &\quad + \frac{1 - \gamma}{\Gamma(\beta)} \sum_{s=a}^t (t - \rho(s))^{\beta-1} f(s) \left[\sum_{k=0}^{\infty} (-1)^k (t-s)^{(k)} \frac{(\nabla^k g)(t)}{k!} \right]. \end{aligned}$$

Since

$$\begin{aligned}(t + \alpha - \sigma(s))^{\alpha-1}(t - s)^{(k)} &= (t + \alpha - \sigma(s))^{\alpha+k+1}, \\ (t - \rho(s))^{\beta-1}(t - s)^{(k)} &= (t + \beta - \sigma(s))^{\beta+k+1},\end{aligned}$$

and $\sum_{s=t-k+1}^t (t - s)^{(k)} = 0$, we have

$$\begin{aligned}(\gamma \diamond_a^{-\alpha, -\beta}(fg))(t) &= \frac{\gamma}{\Gamma(\alpha)} \sum_{k=0}^{\infty} (-1)^k \frac{(\nabla^k g)(t)}{k!} \sum_{s=a}^{t-k} (t + \alpha - \sigma(s))^{\alpha+k-1} f(s) \\ &\quad + \frac{1 - \gamma}{\Gamma(\beta)} \sum_{k=0}^{\infty} (-1)^k \frac{(\nabla^k g)(t)}{k!} \sum_{s=a}^{t-k} (t + \beta - \sigma(s))^{\beta+k-1} f(s).\end{aligned}$$

Because

$$(-1)^k = \frac{\Gamma(-\alpha + 1)\Gamma(\alpha)}{\Gamma(-\alpha + k + 1)\Gamma(k + \alpha)} = \frac{\Gamma(-\beta + 1)\Gamma(\beta)}{\Gamma(-\beta + k + 1)\Gamma(k + \beta)}$$

and $k! = \Gamma(k + 1)$, the above expression becomes

$$\begin{aligned}(\gamma \diamond_a^{-\alpha, -\beta}(fg))(t) &= \frac{\gamma}{\Gamma(\alpha)} \sum_{k=0}^{\infty} (\nabla^k g)(t) \binom{-\alpha}{k} \cdot \left[\frac{1}{\Gamma(k + \alpha)} \sum_{s=a}^{t-k} (t + \alpha - \sigma(s))^{\alpha+k-1} f(s) \right] \\ &\quad + \frac{1 - \gamma}{\Gamma(\beta)} \sum_{k=0}^{\infty} (\nabla^k g)(t) \binom{-\beta}{k} \cdot \left[\frac{1}{\Gamma(k + \beta)} \sum_{s=a}^{t-k} (t + \beta - \sigma(s))^{\beta+k-1} f(s) \right] \\ &= \gamma \sum_{k=0}^{\infty} \binom{-\alpha}{k} (\nabla^k g)(t) (\Delta_a^{-(\alpha+k)} f)(t + \alpha + k) \\ &\quad + (1 - \gamma) \sum_{k=0}^{\infty} \binom{-\beta}{k} (\nabla^k g)(t) (\Delta_a^{-(\beta+k)} f)(t + \beta + k).\end{aligned}$$

□

Remark 155. Choosing $\gamma = 0$ in our Leibniz formula (6.2), we obtain that

$$(\nabla_a^{-\beta}(fg))(t) = \sum_{k=0}^{\infty} \binom{-\beta}{k} [(\nabla^k g)(t)] [(\Delta_a^{-(\beta+k)} f)(t + \beta + k)].$$

Remark 156. Choosing $\gamma = 1$ in our Leibniz formula (6.2), we obtain that

$$(\Delta_a^{-\alpha}(fg))(t + \alpha) = \sum_{k=0}^{\infty} \binom{-\alpha}{k} [(\nabla^k g)(t)] [(\Delta_a^{-(\alpha+k)} f)(t + \alpha + k)]. \quad (6.3)$$

As a particular case of (6.3), let $a = 0$. Then, recalling Lemma 141, we obtain the Leibniz formulas of [23].

The preliminary results presented here, about this new fractional operator, are published in [36] and were presented by the author at the NSC'10, 3rd Conference on Nonlinear Science and Complexity, Ankara, Turkey, July 28–31, 2010.

Appendix A

Maxima code used in Chapter 3

The following Maxima code implements Theorem 59. Examples illustrating the use of our procedure `extremal` below are found in Section 3.4.

```
kill(all)$ ratprint:false$ simpsum:true$ tlimswitch:true$
```

```
sigma(t):=t+1$
```

```
rho(t):=t-1$
```

```
rho2(t):=rho(rho(t))$
```

```
Delta(exp,t):=block( define(f12(t),exp),  
return((f12(sigma(t))-f12(t))) )$
```

```
p(x,y):=(gamma(x+1))/(gamma(x+1-y))$
```

```
SumL(a,t,nu,exp):=block( define(f1(x),exp),  
f1(t)+nu/gamma(nu+1)*sum((p(t+nu-sigma(r),nu-1))*f1(r),r,(a),(t-1))  
)$
```

```
SumR(t1,b,nu1,exp1):=block( define(f2(x),exp1),  
f2(t1)+nu1/gamma(nu1+1)*sum((p(s+nu1-sigma(t1),nu1-1))*f2(s),s,(t1+1),b)  
)$
```

```
DeltaL(a2,t2,alpha2,exp2):=block(
[alpha1:ratsimp(alpha2),a:ratsimp(a2),t:ratsimp(t2)],
define(f3(x),exp2), define(q(x),SumL(a,x,1-alpha1,f3(x))), q0:q(o),
o4:float(ev(ratsimp(Delta(q0,o)),nouns)), o41:subst(o=t,o4),
remfunction(f), remfunction(q), return(o41) )$
```

```
DeltaR(t3,b,alpha3,exp3):=block(
[alpha1:ratsimp(alpha3),b:ratsimp(b),t:ratsimp(t3)],
define(f4(x),exp3), define(q1(o),(SumR(x,b,1-alpha1,f4(x)))),
q10:q1(z),
o5:float(ev(ratsimp(-Delta((SumR(x,b,1-alpha1,f4(x))),x)),nouns)),
o51:subst(x=t,o5), remfunction(f), remfunction(q1), return(o51) )$
```

```
EL(exp7,a,b,alpha7,beta7):=block(
[a:ratsimp(a),b:ratsimp(b),alpha:ratsimp(alpha7),beta:ratsimp(beta7)],
define(LL(t,u,v,w),exp7), b1:diff(LL(t,u,v,w),u),
sa:subst([u=y(sigma(t)),v=DeltaL(a,t,alpha,y(o)),
w=DeltaR(t,b,beta,y(o))],b1),
b2:diff(LL(t,u,v,w),v),
sb:subst([t=x,u=y(sigma(x)),v=DeltaL(a,x,alpha,y(x)),
w=DeltaR(x,b,beta,y(x))],b2),
sb1:DeltaR(o,rho(b),alpha,sb), sb11:subst(o=t,sb1),
b3:diff(LL(t,u,v,w),w),
sc:subst([t=x,u=y(sigma(x)),v=DeltaL(a,x,alpha,y(x)),
w=DeltaR(x,b,beta,y(x))],b3),
sc2:DeltaL(a,p2,beta,sc), sc22:subst(p2=t,sc2), return(sa+sb11+sc22)
)$
```

```
ELt(exp8,a,b,alpha8,beta8,t8):=
ratsimp(subst(t=t8,EL(exp8,a,b,alpha8,beta8)))$
```

```
extremal(L,a,b,A9,B9,alpha9,beta9):=block(
[a:ratsimp(a),b:ratsimp(b),alpha:ratsimp(alpha9),
beta:ratsimp(beta9),A1:ratsimp(A9), B1:ratsimp(B9)],
```

```

eqs:makelist(ratsimp(ELt(L,a,b,alpha,beta,a+i)),i,0,ratsimp((rho2(b)-a))),
vars:makelist(y(ratsimp(a+i)),i,1,ratsimp((rho(b)-a))),
Xi:[a],Xf:[b],Yi:[A1],Yf:[B1],
X:makelist(ratsimp(i),i,1,ratsimp((rho(b)-a))), X:append(Xi,X,Xf),
sols:algsys(subst([y(a)=A1,y(b)=B1],eqs),vars),
Y:makelist(rhs(sols[1][i]),i,1,ratsimp((rho(b)-a))),
Y:append(Yi,Y,Yf),
return(makegamma(ratsimp(minfactorial(makefact(sols[1]))))) )$

```


Appendix B

Maxima code used in Chapter 4

The following Maxima code implements Theorem 88 and Theorem 91.

```
kill(all)$remfunction(all)$remvalue(all)$
fpprec:100$load("draw")$
load("eval_string")$
ratprint:false$
simpsum:true$
simpproduct:true$
tlimswitch:true$

sigma(t,h):=t+h$

rho(t,h):=t-h$

rho2(t,h):=rho(rho(t,h),h)$

Delta(exp,t,h):=block(
define(f12(t),exp),
return((f12(sigma(t,h))-f12(t))/h)
)$

p(x,y,h):=h^y*(((gamma(x/h+1)))/(gamma(x/h+1-y))))$

intHZ(f,var,a9,b9,h9):=block(
```

```
[a:ratsimp(a9),b:ratsimp(b9),h:ratsimp(h9)],
p1:subst(var=k*h,f),
res:sum(p1*h, k, a/h, rho(b,h)/h),
return(res)
)$
```

```
SumL(a,t,nu,exp9,h0):=block(
define(f1(x),exp9),
li:ratsimp(a/h0),
ls:ratsimp((t-h0)/h0),
l1:float((nu/gamma(nu+1))),
(h^nu)*f1(t)+l1*sum((p(t+nu*h0-sigma(r*h0,h0),nu-1,h0))*f1(r*h0),r,li,ls)*h0)$
```

```
SumR(t,b,nu1,exp1,h1):=block(
define(f2(x),exp1),
l1:((nu1/gamma(nu1+1))),
li:ratsimp(sigma(t,h1)/h1),
ls:ratsimp(b/h1),
(h1^nu1)*f2(t)+l1*sum((p(c*h1+nu1*h1-sigma(t,h1),nu1-1,h1))*f2(c*h1),c,li,ls)*h1)$
```

```
DeltaL(a2,t2,alpha2,exp2,h):=block(
[alpha1:ratsimp(alpha2),a:ratsimp(a2),t:ratsimp(t2)],
define(f3(x),exp2),
define(q(x),SumL(a,x,1-alpha1,f3(x),h)),
q0:q(o),
o4:(ev((Delta(q0,o,h)),nouns)),
o41:subst(o=t,o4),
remfunction(f3),
remfunction(q),
return(o41)
)$
```

```
DeltaR(t3,b,alpha3,exp3,h3):=block(
[alpha1:ratsimp(alpha3),h:ratsimp(h3),b:ratsimp(b),t:ratsimp(t3)],
```

```

define(f4(x),exp3),
define(q1(x),(SumR(x,b,1-alpha1,f4(x),h))),
q01:q1(o),
o5:-(ev((Delta(q01,o,h)),nouns)),
o51:subst(o=t,o5),
remfunction(f4),
remfunction(q1),
return(o51))$

EL(exp7,a,b,alpha7,beta7,h7):=block(
[a:ratsimp(a),b:ratsimp(b),alpha:ratsimp(alpha7),beta:ratsimp(beta7),
h:ratsimp(h7)],

define(LL(t,u,v,w),exp7),
b1:diff(LL(t,u,v,w),u),
sa:subst([u=y(sigma(t,h)),v=DeltaL(a,t,alpha,y(t),h),
w=DeltaR(t,b,beta,y(t),h)],b1),
b2:diff(LL(t,u,v,w),v),
sb:subst([t=x,u=y(sigma(t,h)),v=DeltaL(a,x,alpha,y(x),h),
w=DeltaR(x,b,beta,y(x),h)],b2),
sb1:DeltaR(o,rho(b,h),alpha,sb,h),
sb11:subst(o=t,sb1),
b3:diff(LL(t,u,v,w),w),
sc:subst([u=y(sigma(t,h)),v=DeltaL(a,x,alpha,y(x),h),
w=DeltaR(x,b,beta,y(x),h)],b3),
sc2:DeltaL(a,p2,beta,sc,h),
sc22:subst(p2=t,sc2),
return(sa+sb11+sc22)
)$

ELt(exp8,a,b,alpha8,beta8,t8,h):=
ratsimp(subst(t=t8,EL(exp8,a,b,alpha8,beta8,h)))$

extremal2(L,a,b,A9,B9,alpha9,beta9,h9):=block(
[a:ratsimp(a),b:ratsimp(b),h:ratsimp(h9),alpha:ratsimp(alpha9),
beta:ratsimp(beta9),A1:ratsimp(A9),

```

```

B1:ratsimp(B9)],
eqs:makelist(
(ELt(L,a,b,alpha,beta,a+i*h,h)),i,0,ratsimp((rho2(b,h)-a)/h)
),
vars:makelist(y(ratsimp(a+i*h)),i,1,ratsimp((rho(b,h)-a)/h)),
Xi:[a],Xf:[b],Yi:[A1],Yf:[B1],
X:makelist(ratsimp(i*h),i,1,ratsimp((rho(b,h)-a)/h)),
X:append(Xi,X,Xf),
eqs01:subst([y(a)=A1,y(b)=B1],eqs),
eqs1:((eqs01)),
return(eqs1)
)$

extremal(L,a,b,A9,B9,alpha9,beta9,h9):=block(
[a:ratsimp(a),b:ratsimp(b),h:ratsimp(h9),alpha:ratsimp(alpha9),
beta:ratsimp(beta9),A1:ratsimp(A9),
B1:ratsimp(B9)],
eqs:makelist((ELt(L,a,b,alpha,beta,a+i*h,h)),i,0,ratsimp((rho2(b,h)-a)/h)),
vars:makelist(y(ratsimp(a+i*h)),i,1,ratsimp((rho(b,h)-a)/h)),
Xi:[a],Xf:[b],Yi:[A1],Yf:[B1],
X:makelist(ratsimp(i*h),i,1,ratsimp((rho(b,h)-a)/h)),
X:append(Xi,X,Xf),
eqs01:subst([y(a)=A1,y(b)=B1],eqs),
eqs1:((eqs01)),
sols:algsys(eqs1,vars),
if length(sols)=1 then
(Y:makelist(rhs(sols[1][i]),i,1,ratsimp((rho(b,h)-a)/h)),
Y:append(Yi,Y,Yf)
),
disp(float(sols)),
return(makegamma(ratsimp(minfactorial(makefact(sols))))))
)$

LegEqH(L,a,b,alpha,beta,h):=block(
mu:1-alpha,

```

```

nu:1-beta,
duu:diff(L,u,2),
duv:diff(L,u,1,v,1),
duw:diff(L,u,1,w,1),
dvv:diff(L,v,2),
dvw:diff(L,v,1,w,1),
dww:diff(L,w,2),
part1:(h^2)*subst([u=y(sigma(t,h)),v=DeltaL(a,t,alpha,y(x),h),
                    w=DeltaR(t,b,beta,y(x),h)],duu),
part2:2*h^(mu+1)*subst([u=y(sigma(t,h)),v=DeltaL(a,t,alpha,y(x),h),
                        w=DeltaR(t,b,beta,y(x),h)],duv),
part3:(h^mu)*subst([u=y(sigma(t,h)),v=DeltaL(a,t,alpha,y(x),h),
                    w=DeltaR(t,b,beta,y(x),h)],dvv),
part4:subst(t=sigma(t,h),subst([u=y(sigma(t,h)),v=DeltaL(a,t,alpha,y(x),h),
                                w=DeltaR(t,b,beta,y(x),h)],dvv))*(mu*h^mu-h^mu)^2,
part5:intHZ(h^3*subst(t=s,subst([u=y(sigma(t,h)),v=DeltaL(a,t,alpha,y(x),h),
                                w=DeltaR(t,b,beta,y(x),h)],dvv))*((1/gamma(mu+1))*mu*(mu-1)
                                *p(s+mu*h-sigma(sigma(t,h),h),mu-2,h))^2,s,sigma(sigma(t,h),h),b,h),
part6:2*h^(nu+1)*subst([u=y(sigma(t,h)),v=DeltaL(a,t,alpha,y(x),h),
                        w=DeltaR(t,b,beta,y(x),h)],duw)*(nu-1),
part7:2*h^(mu+nu)*subst([u=y(sigma(t,h)),v=DeltaL(a,t,alpha,y(x),h),
                          w=DeltaR(t,b,beta,y(x),h)],dvw)*(nu-1),
part8:2*h^(mu+nu)*subst(t=sigma(t,h),subst([u=y(sigma(t,h)),
        v=DeltaL(a,t,alpha,y(x),h),w=DeltaR(t,b,beta,y(x),h)],dvw))*(mu-1),
part9:h^(2*nu)*subst([u=y(sigma(t,h)),
        v=DeltaL(a,t,alpha,y(x),h),w=DeltaR(t,b,beta,y(x),h)],dww)*(1-nu)^2,
part10:h^(2*nu)*subst(t=sigma(t,h),subst([u=y(sigma(t,h)),
        v=DeltaL(a,t,alpha,y(x),h),w=DeltaR(t,b,beta,y(x),h)],dww)),
part11:intHZ(h^3*subst(t=s,subst([u=y(sigma(t,h)),v=DeltaL(a,t,alpha,y(x),h),
                                w=DeltaR(t,b,beta,y(x),h)],dww))*((1/gamma(nu+1))*nu*(1-nu)
                                *p(t+nu*h-sigma(s,h),nu-2,h))^2,s,a,t,h),
return(part1+part2+part3+part4+part5+part6+part7+part8+part9+part10+part11)
)$

```

```
LegEqMinH(L,a,b,A,B,alpha,beta,h):=block(
```

```

vec: [],
vec: append(vec, [subst([t=a, y(a)=A], ratsimp(LegEqH(L, a, b, alpha, beta, h)))]),
vectTemp: [a],
for i:1 thru length(X)-3 do(
t1:ratsimp(a+i*h),
vectTemp:append(vectTemp, [t1]),
res:subst([t=ratsimp(t1)], ratsimp(LegEqH(L, a, b, alpha, beta, h))),
vec:append(vec, [res])
),
M:matrix(vectTemp, vec),
return(M)
)$

```

```

LeqEqMinAllH(L, a, b, A, B, alpha, beta, sols, h):=block(
a:ratsimp(a), b:ratsimp(b),
Validos: [],
for i:1 thru length(sols) do(
M:LegEqMinH(L, a, b, A, B, alpha, beta, h),
M:subst([y(a)=A, y(b)=B], M),
M:subst(makelist(sols[i][k], k, 1, length(sols[i])), M),
Sinal: [],
c:0,
for l:1 thru matrix_size(M)[2] do (
Sinal:append(Sinal, [is(M[2][l]>=0)]),
if is(M[2][l]>=0)=true then(c:c+1)
),
if c=matrix_size(M)[2] then (
Validos:append(Validos, [i])
)
),
disp(Validos)
)
$

```

```

TESTE(L, a, b, A, B, alpha, beta, h, sols):=block(

```

```

a:ratsimp(a),b:ratsimp(b),
indices:[],
valor:[],
for i:1 thru length(sols) do(
f00:subst(
[u=y(sigma(t,h)),v=DeltaL(a,t,alpha,y(t),h),w=DeltaR(t,b,beta,y(t),h)],L
),
r00:ratsimp(intHZ(f00,t,a,b,h)),
r01:subst(sols[i],r00),
r02:subst([y(a)=A,y(b)=B],r01),
indices:append(indices,[i]),
valor:append(valor,[r02])
),
Mat:matrix(indices,valor),
return(Mat)
)$

```


References

- [1] T. Abdeljawad, F. Jarad and D. Baleanu, Variational optimal-control problems with delayed arguments on time scales, *Adv. Difference Equ.* **2009**, Art. ID 840386, 15 pp. MR2577651
- [2] R. Agarwal, M. Bohner, D. O'Regan and A. Peterson, Dynamic equations on time scales: a survey, *J. Comput. Appl. Math.* **141** (2002), no. 1-2, 1–26. MR1908825
- [3] O. P. Agrawal, Formulation of Euler-Lagrange equations for fractional variational problems, *J. Math. Anal. Appl.* **272** (2002), no. 1, 368–379. MR1930721
- [4] O. P. Agrawal, A general formulation and solution scheme for fractional optimal control problems, *Nonlinear Dynam.* **38** (2004), no. 1-4, 323–337. MR2112177
- [5] O. P. Agrawal, Fractional variational calculus and the transversality conditions, *J. Phys. A* **39** (2006), no. 33, 10375–10384. MR2256598
- [6] O. P. Agrawal, A general finite element formulation for fractional variational problems, *J. Math. Anal. Appl.* **337** (2008), no. 1, 1–12. MR2356049
- [7] O. P. Agrawal and D. Baleanu, A Hamiltonian formulation and a direct numerical scheme for fractional optimal control problems, *J. Vib. Control* **13** (2007), no. 9-10, 1269–1281. MR2356715
- [8] O. P. Agrawal, J. A. Tenreiro Machado and J. Sabatier, Introduction [Special issue on fractional derivatives and their applications], *Nonlinear Dynam.* **38** (2004), no. 1-4, 1–2. MR2112165
- [9] E. Akin, Cauchy functions for dynamic equations on a measure chain, *J. Math. Anal. Appl.* **267** (2002), no. 1, 97–115. MR1886819

-
- [10] R. Almeida and A. B. Malinowska and D. F. M. Torres, A fractional calculus of variations for multiple integrals with application to vibrating string, *J. Math. Phys.* **51** (2010), no. 3, 033503, 12 pp. MR2647882
- [11] R. Almeida, S. Pooseh and D. F. M. Torres, Fractional variational problems depending on indefinite integrals, *Nonlinear Analysis* **75** (2012), no. 3, 1009–1025.
- [12] R. Almeida and D. F. M. Torres, Hölderian variational problems subject to integral constraints, *J. Math. Anal. Appl.* **359** (2009), no. 2, 674–681. MR2546784
- [13] R. Almeida and D. F. M. Torres, Isoperimetric problems on time scales with nabla derivatives, *J. Vib. Control* **15** (2009), no. 6, 951–958. MR2528200
- [14] R. Almeida and D. F. M. Torres, Calculus of variations with fractional derivatives and fractional integrals, *Appl. Math. Lett.* **22** (2009), no. 12, 1816–1820. MR2558546
- [15] R. Almeida and D. F. M. Torres, Fractional variational calculus for nondifferentiable functions, *Comput. Math. Appl.* **61** (2011), no. 10, 3097–3104. MR2799834
- [16] M. R. S. Ammi, R. A. C. Ferreira and D. F. M. Torres, Diamond- α Jensen's inequality on time scales, *J. Inequal. Appl.* **2008**, Art. ID 576876, 13 pp. MR2410768
- [17] G. A. Anastassiou, *Discrete fractional calculus and inequalities*, 2009.
- [18] G. A. Anastassiou, Principles of delta fractional calculus on time scales and inequalities, *Math. Comput. Modelling* **52** (2010), no. 3-4, 556–566. MR2658507
- [19] D. R. Anderson, Taylor polynomials for nabla dynamic equations on time scales, *Panamer. Math. J.* **12** (2002), no. 4, 17–27. MR1941427
- [20] F. M. Atici and P. W. Eloe, A transform method in discrete fractional calculus, *Int. J. Difference Equ.* **2** (2007), no. 2, 165–176. MR2493595
- [21] F. M. Atici and P. W. Eloe, Initial value problems in discrete fractional calculus, *Proc. Amer. Math. Soc.* **137** (2009), no. 3, 981–989. MR2457438
- [22] F. M. Atici and P. W. Eloe, Discrete fractional calculus with the nabla operator, *Electron. J. Qual. Theory Differ. Equ.* **2009**, Special Edition I, No. 3, 12 pp. MR2558828

REFERENCES

- [23] F. M. Atici and S. Şengül, Modeling with fractional difference equations, *J. Math. Anal. Appl.* **369** (2010), no. 1, 1–9. MR2643839
- [24] B. Aulbach and S. Hilger, A unified approach to continuous and discrete dynamics, in *Qualitative theory of differential equations (Szeged, 1988)*, 37–56, Colloq. Math. Soc. János Bolyai, 53 North-Holland, Amsterdam. MR1062633
- [25] D. Baleanu, New applications of fractional variational principles, *Rep. Math. Phys.* **61** (2008), no. 2, 199–206. MR2424086
- [26] D. Baleanu, O. Defterli and O. P. Agrawal, A central difference numerical scheme for fractional optimal control problems, *J. Vib. Control* **15** (2009), no. 4, 583–597. MR2519151
- [27] D. Baleanu and F. Jarad, Discrete variational principles for higher-order Lagrangians, *Nuovo Cimento Soc. Ital. Fis. B* **120** (2005), no. 9, 931–938. MR2189608
- [28] D. Baleanu and F. Jarad, Difference discrete variational principles, in *Mathematical analysis and applications*, 20–29, AIP Conf. Proc., 835 Amer. Inst. Phys., Melville, NY, 2006. MR2258080
- [29] D. Baleanu, T. Maaraba and F. Jarad, Fractional variational principles with delay, *J. Phys. A* **41** (2008), no. 31, 315403, 8 pp. MR2425823
- [30] D. Baleanu and S. I. Muslih, Nonconservative systems within fractional generalized derivatives, *J. Vib. Control* **14** (2008), no. 9-10, 1301–1311. MR2463066
- [31] D. Baleanu, S. I. Muslih and E. M. Rabei, On fractional Euler-Lagrange and Hamilton equations and the fractional generalization of total time derivative, *Nonlinear Dynam.* **53** (2008), no. 1-2, 67–74. MR2411429
- [32] Z. Bartosiewicz and D. F. M. Torres, Noether’s theorem on time scales, *J. Math. Anal. Appl.* **342** (2008), no. 2, 1220–1226. MR2445270
- [33] N. R. O. Bastos, R. A. C. Ferreira and D. F. M. Torres, Necessary optimality conditions for fractional difference problems of the calculus of variations, *Discrete Contin. Dyn. Syst.* **29** (2011), no. 2, 417–437. MR2728463

-
- [34] N. R. O. Bastos, R. A. C. Ferreira and D. F. M. Torres, Discrete-time fractional variational problems. *Signal Processing*, vol. 91 (2011), 513–524.
- [35] N. R. O. Bastos, D. Mozyrska and D. F. M. Torres, Fractional derivatives and integrals on time scales via the inverse generalized Laplace transform, *Int. J. Math. Comput.* **11** (2011), J11, 1–9. MR2800417
- [36] N. R. O. Bastos and D. F. M. Torres, Combined Delta-Nabla Sum Operator in Discrete Fractional Calculus. *Commun. Frac. Calc.* 1 (2010) 41–47.
- [37] M. Bohner, Calculus of variations on time scales, *Dynam. Systems Appl.* **13** (2004), no. 3-4, 339–349. MR2106410
- [38] M. J. Bohner, R. A. C. Ferreira and D. F. M. Torres, Integral inequalities and their applications to the calculus of variations on time scales, *Math. Inequal. Appl.* **13** (2010), no. 3, 511–522. MR2662835
- [39] M. Bohner and G. Sh. Guseinov, The convolution on time scales, *Abstr. Appl. Anal.* **2007**, Art. ID 58373, 24 pp. MR2320804
- [40] M. Bohner and D. A. Lutz, Asymptotic expansions and analytic dynamic equations, *ZAMM Z. Angew. Math. Mech.* **86** (2006), no. 1, 37–45. MR2193645
- [41] M. Bohner and A. Peterson, *Dynamic equations on time scales: an introduction with applications.*, Birkhäuser, Boston, MA, 2001. MR1843232
- [42] M. Bohner and A. Peterson, Laplace transform and Z -transform: unification and extension, *Methods Appl. Anal.* **9** (2002), no. 1, 151–157. MR1948468
- [43] M. Bohner and A. Peterson, *Advances in dynamic equations on time scales*, Birkhäuser Boston, Boston, MA, 2003. MR1962542
- [44] A. M. C. Brito da Cruz, N. Martins, D. F. M. Torres, Higher-order Hahn’s quantum variational calculus, *Nonlinear Analysis* 75 (2012), no. 3, 1147–1157.
- [45] F. Chen, X. Luo and Y. Zhou, Existence results for nonlinear fractional difference equation, *Adv. Difference Equ.* **2011**, Art. ID 713201, 12 pp. MR2747089

- [46] J. Cresson, G. S. F. Frederico and D. F. M. Torres, Constants of motion for non-differentiable quantum variational problems, *Topol. Methods Nonlinear Anal.* **33** (2009), no. 2, 217–231. MR2549615
- [47] J. M. Davis, I. A. Gravagne, B. J. Jackson and R. J. Marks, The Laplace transform on time scales revisited, *J. Math. Anal. Appl.* **332** (2007), no. 2, 1291–1307. MR2324337
- [48] J. B. Díaz and T. J. Osler, Differences of fractional order, *Math. Comp.* **28** (1974), 185–202. MR0346352
- [49] R. A. El-Nabulsi and D. F. M. Torres, Necessary optimality conditions for fractional action-like integrals of variational calculus with Riemann-Liouville derivatives of order (α, β) , *Math. Methods Appl. Sci.* **30** (2007), no. 15, 1931–1939. MR2349759
- [50] R. A. El-Nabulsi and D. F. M. Torres, Fractional actionlike variational problems, *J. Math. Phys.* **49** (2008), no. 5, 053521, 7 pp. MR2421931
- [51] R. A. C. Ferreira, *Calculus of Variations on Time Scales and Discrete Fractional Calculus*, PhD thesis, University of Aveiro, 2010.
- [52] R. A. C. Ferreira and D. F. M. Torres, Higher-order calculus of variations on time scales, in *Mathematical control theory and finance*, 149–159, Springer, Berlin, 2008. MR2484109
- [53] R. A. C. Ferreira and D. F. M. Torres, Fractional h -difference equations arising from the calculus of variations, *Appl. Anal. Discrete Math.* **5** (2011), no. 1, 110–121. MR2809039
- [54] T. Fort, The calculus of variations applied to Nörlund’s sum, *Bull. Amer. Math. Soc.* **43** (1937), no. 12, 885–887. MR1563653
- [55] G. S. F. Frederico and D. F. M. Torres, A formulation of Noether’s theorem for fractional problems of the calculus of variations, *J. Math. Anal. Appl.* **334** (2007), no. 2, 834–846. MR2338631
- [56] G. S. F. Frederico and D. F. M. Torres, Fractional conservation laws in optimal control theory, *Nonlinear Dynam.* **53** (2008), no. 3, 215–222. MR2433010

-
- [57] G. S. F. Frederico and D. F. M. Torres, Fractional Noether's theorem in the Riesz-Caputo sense, *Appl. Math. Comput.* **217** (2010), no. 3, 1023–1033. MR2727141
- [58] C. S. Goodrich, Existence of a positive solution to a class of fractional differential equations, *Appl. Math. Lett.* **23** (2010), no. 9, 1050–1055. MR2659137
- [59] C. S. Goodrich, Continuity of solutions to discrete fractional initial value problems, *Comput. Math. Appl.* **59** (2010), no. 11, 3489–3499. MR2646320
- [60] R. Hilfer (Ed.): *Applications of fractional calculus in physics*. World Scientific Publishing Co., Inc., River Edge, NJ, 2000. viii+463 pp. ISBN: 981-02-3457-0. MR1890104
- [61] S. Hilger, Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten. PhD Thesis, Universität Würzburg, 1988.
- [62] S. Hilger, Analysis on measure chains—a unified approach to continuous and discrete calculus, *Results Math.* **18** (1990), no. 1-2, 18–56. MR1066641
- [63] S. Hilger, Differential and difference calculus—unified!, *Nonlinear Anal.* **30** (1997), no. 5, 2683–2694. MR1602920
- [64] R. Hilscher and V. Zeidan, Nonnegativity of a discrete quadratic functional in terms of the (strengthened) Legendre and Jacobi conditions, *Comput. Math. Appl.* **45** (2003), no. 6-9, 1369–1383. MR2000603
- [65] R. Hilscher and V. Zeidan, Calculus of variations on time scales: weak local piecewise C_{rd}^1 solutions with variable endpoints, *J. Math. Anal. Appl.* **289** (2004), no. 1, 143–166. MR2020533
- [66] R. Hilscher and V. Zeidan, Nonnegativity and positivity of quadratic functionals in discrete calculus of variations: survey, *J. Difference Equ. Appl.* **11** (2005), no. 9, 857–875. MR2159802
- [67] M. Holm, The theory of discrete fractional calculus: Development and application, PhD thesis, 2010.
- [68] F. Jarad and D. Baleanu, Discrete variational principles for Lagrangians linear in velocities, *Rep. Math. Phys.* **59** (2007), no. 1, 33–43. MR2308632

- [69] F. Jarad, D. Baleanu and T. Maraaba, Hamiltonian formulation of singular Lagrangians on time scales, *Chinese Phys. Lett.* **25** (2008), no. 5, 1720–1723.
- [70] V. Kac, P. Cheung, *Quantum Calculus*, Springer-Verlag, New York, 2002. MR1865777
- [71] W. G. Kelley and A. C. Peterson, *Difference equations*, Academic Press, Boston, MA, 1991. MR1142573
- [72] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006. MR2218073
- [73] M. Klimek, Stationarity-conservation laws for fractional differential equations with variable coefficients, *J. Phys. A* **35** (2002), no. 31, 6675–6693. MR1928856
- [74] B. Kuttner, On differences of fractional order, *Proc. London Math. Soc.* (3) **7** (1957), 453–466. MR0094618
- [75] V. Lakshmikantham, S. Sivasundaram and B. Kaymakçalan, *Dynamic systems on measure chains*, Mathematics and its Applications, 370, Kluwer Acad. Publ., Dordrecht, 1996. MR1419803
- [76] A. B. Malinowska and D. F. M. Torres, Strong minimizers of the calculus of variations on time scales and the Weierstrass condition, *Proc. Est. Acad. Sci.* **58** (2009), no. 4, 205–212. MR2604248
- [77] A. B. Malinowska and D. F. M. Torres, On the diamond-alpha Riemann integral and mean value theorems on time scales, *Dynam. Systems Appl.* **18** (2009), no. 3-4, 469–481. MR2562284
- [78] A.B. Malinowska and D.F.M. Torres, Fractional variational calculus in terms of a combined Caputo derivative, IFAC Workshop on Fractional Derivative and Applications (IFAC FDA'2010), University of Extremadura, Badajoz, Spain, October 18-20, 2010.
- [79] A. B. Malinowska and D. F. M. Torres, Natural boundary conditions in the calculus of variations, *Math. Methods Appl. Sci.* **33** (2010), no. 14, 1712–1722. MR2723491
- [80] A. B. Malinowska and D. F. M. Torres, Generalized natural boundary conditions for fractional variational problems in terms of the Caputo derivative, *Comput. Math. Appl.* **59** (2010), no. 9, 3110–3116. MR2610543

-
- [81] A. B. Malinowska and D. F. M. Torres, Leitmann's direct method of optimization for absolute extrema of certain problems of the calculus of variations on time scales, *Appl. Math. Comput.* **217** (2010), no. 3, 1158–1162. MR2727155
- [82] A. B. Malinowska and D. F. M. Torres, The Hahn quantum variational calculus, *J. Optim. Theory Appl.* **147** (2010), no. 3, 419–442. MR2733985
- [83] A. B. Malinowska and D. F. M. Torres, Fractional calculus of variations for a combined Caputo derivative, *Fract. Calc. Appl. Anal.*, Vol. 14, No 4 (2011), pp. 523–537.
- [84] N. Martins and D. F. M. Torres, Calculus of variations on time scales with nabla derivatives, *Nonlinear Anal.* **71** (2009), no. 12, e763–e773. MR2671876
- [85] N. Martins and D. F. M. Torres, Generalizing the variational theory on time scales to include the delta indefinite integral, *Comput. Math. Appl.* **61** (2011), no. 9, 2424–2435. MR2794990
- [86] K. S. Miller and B. Ross, Fractional difference calculus, in *Univalent functions, fractional calculus, and their applications (Kōriyama, 1988)*, 139–152, Ellis Horwood Ser. Math. Appl, Horwood, Chichester, 1989. MR1199147
- [87] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, A Wiley-Interscience Publication, Wiley, New York, 1993. MR1219954
- [88] D. Mozyrska and D. F. M. Torres, A study of diamond-alpha dynamic equations on regular time scales, *Afr. Diaspora J. Math. (N.S.)* **8** (2009), no. 1, 35–47. MR2500193
- [89] D. Mozyrska and D. F. M. Torres, Minimal modified energy control for fractional linear control systems with the Caputo derivative, *Carpathian J. Math.* **26** (2010), no. 2, 210–221. MR2721980
- [90] D. Mozyrska and D. F. M. Torres, Modified Optimal Energy and Initial Memory of Fractional Continuous-Time Linear Systems, *Signal Process.* **91** (2011), no. 3, 379–385.
- [91] S. I. Muslih and D. Baleanu, Fractional Euler-Lagrange equations of motion in fractional space, *J. Vib. Control* **13** (2007), no. 9-10, 1209–1216. MR2356711

- [92] T. Odziejewicz, A. B. Malinowska and Delfim F. M. Torres, Fractional variational calculus with classical and combined Caputo derivatives, *Nonlinear Analysis* **75** (2012), no. 3, 1507–1515.
- [93] T. Odziejewicz and D. F. M. Torres, Fractional calculus of variations for double integrals, *Balkan J. Geom. Appl.* **16** (2011), no. 2, 102–113. MR2785736
- [94] K. B. Oldham and J. Spanier, *The fractional calculus*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York, 1974. MR0361633
- [95] M. D. Ortigueira, Fractional central differences and derivatives, *J. Vib. Control* **14** (2008), no. 9-10, 1255–1266. MR2463063
- [96] M. D. Ortigueira, The fractional quantum derivative and its integral representations, *Commun. Nonlinear Sci. Numer. Simul.* **15** (2010), no. 4, 956–962. MR2557004
- [97] M. D. Ortigueira, On the fractional linear scale invariant systems, *IEEE Trans. Signal Process.* **58** (2010), no. 12, 6406–6410. MR2790468
- [98] M. D. Ortigueira, *Fractional calculus for scientists and engineers*, Lecture Notes in Electrical Engineering, 84, Springer, Dordrecht, 2011. MR2768178
- [99] M. D. Ortigueira and A. G. Batista, A fractional linear system view of the fractional Brownian motion, *Nonlinear Dynam.* **38** (2004), no. 1-2, 295–303.
- [100] I. Podlubny, *Fractional differential equations*, Mathematics in Science and Engineering, 198, Academic Press, San Diego, CA, 1999. MR1658022
- [101] S. Pooseh, H. S. Rodrigues and Delfim F. M. Torres, Fractional derivatives in Dengue epidemics, *AIP Conf. Proc.* 1389 (2011), 739–742.
- [102] E. M. Rabei, K. I. Nawafleha, R. S. Hijjawia, S. I. Muslihc and D. Baleanu, The Hamilton formalism with fractional derivatives, *J. Math. Anal. Appl.* **327** (2007), no. 2, 891–897. MR2279972
- [103] E. M. Rabei, D. M. Tarawneh, S. I. Muslih and D. Baleanu, Heisenberg’s equations of motion with fractional derivatives, *J. Vib. Control* **13** (2007), no. 9-10, 1239–1247. MR2356713

-
- [104] M. R. S. Rahmat and M. S. Md. Noorani, Fractional integrals and derivatives on time scales with an application, *Comput. Math. Appl.* (2009), DOI: 10.1016/j.camwa.2009.10.013
- [105] F. Riewe, Nonconservative Lagrangian and Hamiltonian mechanics, *Phys. Rev. E* (3) **53** (1996), no. 2, 1890–1899. MR1401316
- [106] F. Riewe, Mechanics with fractional derivatives, *Phys. Rev. E* (3) **55** (1997), no. 3, part B, 3581–3592. MR1438729
- [107] B. Ross, S. G. Samko and E. R. Love, Functions that have no first order derivative might have fractional derivatives of all orders less than one, *Real Anal. Exchange* **20** (1994), 140–157. MR1313679
- [108] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives*, Translated from the 1987 Russian original, Gordon and Breach, Yverdon, 1993. MR1347689
- [109] S. Sengul, *Discrete Fractional Calculus and Its Applications to Tumor Growth*, Master Thesis, 2010.
- [110] M. F. Silva, J. A. Tenreiro Machado and R. S. Barbosa, Using fractional derivatives in joint control of hexapod robots, *J. Vib. Control* **14** (2008), no. 9-10, 1473–1485. MR2463073
- [111] V. Spedding, Taming nature’s numbers, *New Scientist*, 19 July 2003, 28-31.
- [112] D. F. M. Torres, The Variational Calculus on Time Scales, *Int. J. Simul. Multidisci. Des. Optim.* **4** (2010), 11–25.
- [113] D. F. M. Torres and G. Leitmann, Contrasting two transformation-based methods for obtaining absolute extrema, *J. Optim. Theory Appl.* **137** (2008), no. 1, 53–59. MR2386612
- [114] B. van Brunt, *The calculus of variations*, Springer, New York, 2004. MR2004181

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