THE PROCESS OF LEARNING MATHEMATICS: A FUZZY SET APPROACH

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The fuzzy sets theory is used for the description of the process of learning and, as an application, a classroom experiment is presented about learning mathematics.

1. Introduction.

There are often situations in real life in which definitions do not have clear boundaries; for example this happens when we speak about the "long rivers" or "high mountains" of a country, about the "young people" of a town, about the "tall pupils" of a school, e.t.c.. The fuzzy sets theory was created in response to the need to have a mathematical representation of such kind of situations.

Let U denote the universal set of the discourse. Then a fuzzy set A in U, initiated by Zadeh [7], is defined by means of the membership function mA, which assigns to each element of U a real value from the interval [0,1]. More specifically A = {(x,mA(x)) : x Є U} , where mA U ----> [0,1).

The value mA(x), called the membership degree (or grade) of x in A, expresses the degree to which x verifies the characteristic property of A. Thus, the nearer the value mA(x) to 1 , the higher the membership degree of x in A.

It is always said that the formal mathematical knowledge comes from the very exactness of the science of Mathematics and therefore there is no possible space in Mathematics for any lack of definition, or vagueness. But, the knowledge that students have about the various mathematical concepts is usually imperfect, characterized by a different degree of depth. This suggests that the application of the fuzzy sets theory in mathematical education could be a very promising tool in order to represent such situations and get useful conclusions.

2. The model of Voss for the process of learning.

The concept of learning has been fundamental to the study of human cognitive action. But, while everyone in general knows what learning is, the understanding of the nature of this process has proved to be complicated.

Voss [6] has developed an argument that learning is a specific case of the general class of the transfer of knowledge. In adopting this perspective he argues that any instance of acquisition of knowledge involves the use of existing knowledge. Basically, according to his argument, learning consists of successive problem-solving activities, in which the input information is represented of existing knowledge with the solution occurring when the input is appropriately interpreted. The whole process involves the following states:

a) representation of the input data, b) interpretation of these data; c) generalization of the new knowledge and d) categorization of this knowledge.

The representation of the stimulus input is relied upon the individual's ability to use content of his memory in order to find information that will facilitate a solution development.

Learning consists of developing an appropriate number of interpretations of the various input information and generalizing the respective interpretations to a variety of situations. When knowledge becomes substantial, much of the process involves categorization, i.e. the input information is interpreted in terms of the classes of the existing knowledge. Thus the individual becomes able to relate new information to his knowledge structures.

3. A fuzzy set representation of the process of learning.

Let us consider a group G of n students during the process of learning in the classroom, $n \in N$, $n>=2$. Σ

Obviously, from the point of view of the teacher, there exists an uncertainity about the degree of acquisition of each state of the process (as it has been described in the previous paragraph) from his students, a fact which gave us the hint to introduce the fuzzy sets theory in order to achieve a mathematical representation of the process of learning in the classroom.For this, let us denote by Ai, i=1, 2, 3, the state of interpretation, generalization and categorization respectively and by a, b, c, d, e the linguistic labels of negligible, low, intermediate, high and complete acquisition respectively of each of the Ai's. Consider the set U = {a,b,c,d,e} , then we are going to represent the Ai's as fuzzy sets in U.

In fact, if nia, nib, nic, nid and nie denote the numbers of the students that have achieved negligible,low, intermediate, high and complete acquisition of the state Ai respectively, i=1, 2, 3, we can define the membership function mAi by mAi(x)=nix/n, for each x in U and therefore we can write Ai={(x,nix/n): x Є U}.

It becomes clear then that Σ mAi(x)=1.

X Є U,i=l ,2, 3

At this point notice that a fuzzy relationship, like the classical ones, can be considered as a fuzzy set of tuples each one of which posseses a degree of membership included between 0 and 1.

Consider now the fuzzy relation R={(s, mr(s)):s=(x, y, z) Є U3}, where the membership function mR is defined by mR(s)=mAl(x)mA2(y)mA3(z), for all s=(x,c,z) in U3. This definition satisfies the axioms of aggregation operations in fuzzy sets and further we have that Σ mR(s)=1 (see section 6.8 of [3], p.p. 282-286).

s Є U3

The fuzzy relation R represents all the possible profiles (overall states) of the behaviour of a student during the learning process.

In the next, and in order to simplify our notation, we shall write ms instead of mR(s).

Assume now that one wants to study the behaviour of k groups of students during the learning process of the same subject, or the behaviour of the same group of students during the learning process of k different subjects, k Є N, k>=2.

In this case it becomes necessary to inroduce the fuzzy variables Ai(t), where i=1, 2, 3 and t=1,2, ... k.

Then the pseudofrequency f (s) of the overall state s(t) is

k

given by the sum Σ ms(t), while the probability of s(t) is

t=l

given by p(s)=f (s)/ **Σ** f(s), where Σ f(s) denotes the sum of

s Є U3 S Є U3

all pseudofrequences. But, since Σ ms=1, it becomes

S**E**U3

clear that Σ f (s)=k and therefore p(s)=f(s)/k.

s Є U3

Finally the possibility of s(t) is given by r(s)=f(s)/max f(s), where max f(s) denotes the maximal

s Є U3 s Є U3

pseudofrequency. The possibility of s(t) measures the degree of evidence of combined results, i.e. in other words one may say that r(s) gives the "relative probability" of s(t) with respect to the other overall states

4. An application of the fuzzy sets theory to the process of learning mathematics: A classroom experiment.

The following experiment took place at the Technological Educational Institute (T.E.I.) of Mesolonghi (Greece), when I was teaching the definite integral to a group of 35 students of the School of Administration and Economics.

During my 3 hour lecture I used the method of rediscovery keeping in mind what Polya [5] says about active learning: "For an effective learning the learner discovers alone the biggest possible, under the circumstances, part of the new information".

Thus, in my short introduction, I presented the concept of the definite integral through the need of calculating an area under a curve, but I gave the fundamental theorem of integral calculus (connectlng the definite with the indefinite integral of a continuous function in the interval [a, b] without proof. Then I left my students to work alone on their papers and 1 was inspecting their works and reactions, giving to them from time to time suitable hints or instructions. My aim was to help them to understand the basic methods of calculating the definite integrals, with respect to what they already knew about the indefinite integrals (state A1). I observed that 17,8 and 10 students achieved intermediate, high and complete understanding of the new subject respectively (n1a=nlb=0, nic=l7, n1d=8 and nie=l0.

Thus A1= {(a,0), (b,0), (c,17/35), (d,8/35),(e,l0/35)}.

In the next step I gave for solution to the students a number of exercises and simple problems (involving the use of the rules and the basic techniques of integration for the case of definite integrals, calculation of improper integrals as limits of definite integrals, and calculation of the area under a curve or among curves), aiming to help them in generalizing the new information to a variety of situations (state A2).

In the same way 1 found that

A2={(a,6/35), (b,6/35), (c,14/35), (d,9/35),(e,0)}.

At the final step 1 gave to the students for solution a number of composite problems (involving applications to economics, such as present value of cash flows, consumer's and producer's surplus, calculation of probability density functions e.t.c.; see [1], chapter 17) relating the new information to their existing knowledge structures (state A3).

In this case 1 found that

A3 = {(a,12/35),(b ,l0/35), (c,13/35), (d,0),(e,0)}.

A few days later I gave the same lecture to a group of 30 students of another department of the School of Administration and Economics. This time I found that

A1={ (a,0), (b,6/30), (c,15/30), (d,9/30), (e,0)

A2={(a,6/30), (b,8/30), (c,16/30), (d,0), (e,0)}

and A3={(a,12/30), (b,9/30), (c,9/30), (d,0), (e,0)}.

Looking at the Ai's, it can be seen that, for both groups, the higher the i, the lower the second members of the pairs having as first members the higher degrees of acquisition d and e (in all cases except one). This means that the higher the state of learning, the iower the degree of acquisition of it.

Further, on comparing, in the same way, the degrees of acquisition of the states of learning from the two groups, it can be observed that the first group was better in general than the second one.

In table 1 (see below) the probabilities and possibilities are given of the profiles having non zero pseudofrequences. It turns out that the profile s=(c, c, a) had the highest pseudofrequency for the two groups of students of our experiment (ms(l)=mA1 (c)mA2 (c)mA3 (a)=(17/35). (l4/35). (l2/35)=0.067, ms(2)=(l5/30). (l6/30). (12/30)=0.107 and therefore f (s)=0.174). Therefore it had also the highest probabality of occurrence (p(s)=0.174/2=0.087, or 8.7%), while its possibility was 1.

TABLE 1

Probabilities and possibilities of profiles (f (s)=0) of the classroom experiment (section 3)

Ai	A2	A3	ms(1)	ms(2)	f(s)	p(s)	r(s)
b	b	b	O	0.016	0.016	0.008	0.092
b	a	b	O	0.012	0.012	0.006	0.069
b	c	b	0	0.032	0.032	0.016	0.184
b	b	a	O	0.021	0.021	0.010	0.121
b	b	c	O	0.016	0.016	0.008	0.092
b	a	a	O	0.016	0.016	0.008	0.092
b	a	c	O	0.012	0.012	0.006	0.069
b	c	a	O	0.042	0.042	0.021	0.241
b	c	c	O	0.032	0.032	0.016	0.184
c	c	c	0.072	0.080	0.152	0.076	0.874
c	a	c	0.082	0.030	0.112	0.056	0.644
c							
c	b	c	0.031	0.040	0.071	0.036	0.408
c	d	c	0.046	o	0.046	0.023	0.264
c	c	a	0.067	0.107	0.174	0.087	1
c	c	b	0.056	0.008	0.064	0.032	0.368
c	a	a	0.028	0,040	0.068	0.034	0.391
c	a	b	0.024	0.030	0.054	0.027	0.310
c	b	a	0.028	0.053	0.081	0.040	0.466
c	b	b	0.024	0.040	0.064	0.032	0.368
c	d	a	0.043	o	0.043	0.022	0.247
c	d	b	0.036	o	0.036	0.018	0.207
d	d	a	0.020	o	0.020	0.010	0.115
d	d	b	0.017	o	0.017	0.008	0.098
d	d	c	0.022	o	0.022	0.011	0.126
d	a	a	0.013	0.024	0.037	0.018	0.213
d	a	b	0.011	0.018	0.029	0.014	0.167
d	a	c	0.015	0.018	0.033	0.016	0.190
d	b	a	0.013	0.032	0.045	0.022	0.259
d	b	b	0.011	0.024	0.035	0.018	0.201
d	b	c	0.014	0.024	0.038	0.019	0.218
d	c	a	0.031	0.064	0.095	0,048	0.546
d	c	b	0.026	0.048	0.074	0.037	0.425
d	c	c	0.034	0.048	0.082	0.041	0.471
e	a	a	0.017	o	0.017	0.008	0.098
e	a	b	0.014	o	0.014	0.007	0.080
e	a	c	0,018	o	0.018	0.009	0.103
e	b	a	0.017	o	0.017	0.008	0.098
e	b	b	0.014	o	0.014	0.007	0.080
e	b	c	0.018	o	0.018	0.009	0.103
e	c	a	0.039	o	0.039	0.020	0.224
e	c	b	0.033	o	0.033	0.016	0.190
e	c	c	0.042	o	0.042	0.021	0.241
e	d	a	0.025	o	0.025	0.012	0.144
e	d	b	0.021	o	0.021	0.010	0.121
e	d	c	0.027	o	0.027	0.014	0.155

NOTE : In Table 1 we should have Σ ms(i)=l (i=1,2),

S Є U3

Σ f (s)=2, Σ p(s)=1 and Σ r(s)=(Σ f(s))/maxf (s)=2/0.174 =

s Є U3 s Є U3 s Є U3 s Є U3 s Є U3

=11.494, but, because of the roundoff errors appeared, we get slightly different values for the sums above.

5. Remarks and conclusions.

From the experiment of the previous paragraph it becomes evident that the use of the fuzzy sets theory for the mathematical representation of the process of learning, apart from its theoretical interest, leads to useful quantitive conclusions which can give an effective help to the teacher of mathematics in getting a concentrating view of his students abilities and skills and therefore to readapt, according to each case, the process, the rate and the method of his tuition.

It becomes also evident that the same method, with the proper each time modifications, could be used in the classroom during the learning process, not only of mathematics but of many other cognitive subjects.

We note that an analogous application of the fuzzy sets theory has been attempted on the van Hiele model for the teaching of the Euclidian Geometry [4].

Finally we must note that analogous efforts, but with different methodologies of development, have been attempted and by other investigators in the area of mathematical education (e.g. see [2] and its references).

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