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SHAPE ANALYSIS OF AN ADAPTIVE ELASTIC ROD MODEL*

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Abstract. We analyze the shape semiderivative of the solution to an asymptotic nonlinear adaptive elastic rod model, derived in Figueiredo and Trabucho [*Math. Mech. Solids*, 9 (2004), pp. 331–354], with respect to small perturbations of the cross section. The rod model is defined by generalized Bernoulli–Navier elastic equilibrium equations and an ordinary differential equation with respect to time. Taking advantage of the model's special structure and the regularity of its solution, we compute and completely identify, in an appropriate functional space involving time, the weak shape semiderivative.

Key words. adaptive elasticity, rod, shape derivative

AMS subject classifications. 74B20, 74K10, 74L15, 90C31

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1. Introduction.

In this paper we consider a sensitivity analysis problem in shape optimization: the calculus of the derivative of the solution to an asymptotic nonlinear adaptive elastic rod model, with respect to shape variations of the cross section of the rod. More precisely, for each small parameter $s \in [0,\delta]$ we define a perturbed adaptive elastic rod $\overline{\Omega}_s = \overline{\omega}_s \times [0,L]$. The scalar $L > 0$ is its length, and $\omega_s = \omega + s\,\theta(\omega)$ is a perturbation of a fixed cross section ω, in the direction of the vector field $\theta = (\theta_1, \theta_2)$, that realizes the shape variation. To each rod $\overline{\Omega}_s$ we associate the corresponding unique solution (u^s, d^s) of the asymptotic adaptive elastic rod model, derived in Figueiredo and Trabucho [7]. The purpose of this paper is to compute the limit $(\frac{u^s-u}{s}, \frac{d^s-d}{s})$, when $s \to 0^+$, where (u,d) is the solution's rod model for the case $s = 0$. This limit is the semiderivative of the shape function $J : s \in [0,\delta] \to J(\Omega_s) = (u^s, d^s)$, at $s = 0$ in the direction of the vector field θ (in the sense of Delfour and Zolésio [5, p. 289]), or equivalently, the material derivative of the map J at $s = 0$ (in the sense of Haslinger and Mäkinen [8, p. 111]).

The difficulties that arise in the computation of the limit $(\frac{u^s-u}{s}, \frac{d^s-d}{s})$, when $s \to 0^+$, are caused by the complicated form of the asymptotic adaptive elastic rod model derived in Figueiredo and Trabucho [7]. In fact, this is a simplified adaptive elastic model, proposed for the mathematical modeling of the physiological process of bone remodeling. It couples the generalized Bernoulli–Navier elastic equilibrium equations with an ordinary differential equation with respect to time, which is the remodeling rate equation. This latter equation expresses the process of absorption and deposition of bone material due to external stimulus (cf. Cowin and Hegedus [3] and Hegedus and Cowin [9] for a description of the theory of adaptive elasticity, Cowin and Nachlinger [4] for uniqueness results, and Monnier and Trabucho [10] for existence results of three-dimensional solutions). For each $s \in [0,\delta]$, the pair (u^s, d^s) is the unique solution of this asympotic adaptive elastic rod model, where u^s is the displacement vector field

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of the rod $\overline{\Omega}_s$ and d^s is a scalar field that represents the change in volume fraction of the elastic material (from a reference volume fraction) in the rod $\overline{\Omega}_s$. Moreover, u^s is the solution of the generalized Bernoulli–Navier equilibrium equations, and d^s is the solution of the remodeling rate equation. In addition, u^s and d^s are coupled in the model because the material coefficients depend on d^s and the remodeling rate equation depends on u^s.

In spite of this complex structure, we are able to compute the limit $(\frac{u^s-u}{s}, \frac{d^s-d}{s})$ when $s \to 0^+$. There are two main results in this paper, which lead to this limit's computation. The first principal result states that, for each time t, the sequence $(\frac{u^s-u}{s}, \frac{d^s-d}{s})(.,t)$ converges weakly to $(\bar{u}, \bar{d})(.,t)$, when $s \to 0^+$, in an appropriate functional space of Sobolev type. (We denote by (u^s, d^s) the solution of the perturbed rod model, formulated in the unperturbed fixed domain $\overline{\Omega}$, which is the domain of (u, d).) The second main result identifies the weak shape semiderivative denoted by $(\bar{u}, \bar{d})$; it is the unique solution of a nonlinear problem which couples a variational equation (whose solution is $\bar{u}$) and depends on (u, d) and $\bar{d}$, and an ordinary differential equation with respect to time (whose solution is $\bar{d}$) that depends on (u, d) and $\bar{u}$.

The reasonings that we have used to achieve these two results are next summarized. We show that the sequences (u^s, d^s) and $(\frac{u^s-u}{s}, \frac{d^s-d}{s})$ are bounded in appropriate functional spaces, involving time; we use the continuity, the ellipticity, the regularity properties, and the special structure of the asymptotic adaptive elastic rod model. In order to identify the weak shape semiderivative we also apply the weak and/or strong convergence of the sequences $\{u^s\}$ and $\{d^s\}$, when $s \to 0^+$, and again the special structure of the asymptotic adaptive elastic rod model. In particular, due to the form of the remodeling rate equation, we are able to use the integral Gronwall's inequality, which is the key to obtaining the estimates for the sequences $\{d^s\}$ and $\{\frac{d^s-d}{s}\}$ and to identifying the ordinary differential equation with respect to time, whose solution is $\bar{d}$.

Finally let us briefly explain the contents of the paper. After this introduction, in section 2, we describe the problem P_s, which is the asymptotic nonlinear adaptive elastic model for the perturbed rod $\overline{\Omega}_s$; we also prove a regularity property of its solution, and finally we describe the shape problem that we want to solve. In section 3 we reformulate the problem P_s in the unperturbed domain $\overline{\Omega}$; this reformulation is necessary because, in order to compute the limit of the quotient sequence $(\frac{u^s-u}{s}, \frac{d^s-d}{s})$, the vector fields u^s, u, d^s, and d must be defined in the same fixed domain, independent of s. In section 4 we prove that all the sequences $\{u^s\}$, $\{d^s\}$, $\{\frac{u^s-u}{s}\}$, and $\{\frac{d^s-d}{s}\}$ are bounded in appropriate functional spaces involving time; we determine, for each time t, the weak limit of the quotient sequence $(\frac{u^s-u}{s}, \frac{d^s-d}{s})(.,t)$ when $s \to 0^+$; and we identify the weak shape semiderivative (this identification is summarized in theorem 4.11). Finally we present some conclusions and future work.

2. Description of the problem.

In this section we first introduce the notation used in this paper; namely, we consider a family of rods $\overline{\Omega}_s = \overline{\omega}_s \times [0, L]$, with length L and cross section ω_s, parameterized by $s \in [0, \delta]$, which is a small parameter. Next, for each s, we describe the adaptive elastic rod model denoted by P_s, derived by Figueiredo and Trabucho [7]. We prove a regularity result for the displacement vector field u_s, the first component of the solution (u_s, d_s) of P_s. Finally, we describe the shape problem under consideration in this paper.

2.1. Notation.

Let $\delta > 0$ be a small parameter, and for each $s \in [0, \delta]$ we consider the perturbation I_s of the identity operator I in $\mathbb{R}^2$, defined by $I_s(x_1, x_2) = (I + s\theta)(x_1, x_2) = (x_{s1}, x_{s2})$, for all $(x_1, x_2) \in \mathbb{R}^2$, where $\theta = (\theta_1, \theta_2) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a vector field regular enough (at least $\theta \in [W^{2,\infty}(\mathbb{R}^2)]^2$). Let ω be an open, bounded, and connected subset of $\mathbb{R}^2$, with a boundary $\partial\omega$ regular enough. For each $s \in [0,\delta]$ we define $\omega_s = I_s(\omega)$, which is the perturbation of ω in the direction of the vector field θ. We also denote by $\overline{\Omega}_s$ the set occupied by a cylindrical adaptive elastic rod, in its reference configuration, with length $L > 0$ and cross section ω_s, that is, $\overline{\Omega}_s = \overline{\omega}_s \times [0,L] = I_s(\overline{\omega}) \times [0,L] \subset \mathbb{R}^3$. Moreover, we denote by $x_s = (x_{s1}, x_{s2}, x_3)$ a generic element of $\overline{\Omega}_s$ and define the sets $\Gamma_s = \partial\omega_s \times\,]0,L[$, $\Gamma_{s0} = \overline{\omega}_s \times \{0\}$, $\Gamma_{sL} = \overline{\omega}_s \times \{L\}$, and $\Gamma_{s0,L} = \Gamma_{s0} \cup \Gamma_{sL}$, where $\partial\omega_s$ is the boundary of ω_s. These last four sets represent, respectively, the lateral boundary of the rod $\overline{\Omega}_s$ and its extremities. We assume that, for each $s \in [0,\delta]$, the coordinate system (O, x_{s1}, x_{s2}, x_3) is a principal system of inertia associated with the rod Ω_s. Consequently, axis Ox_3 passes through the centroid of each section $\omega_s \times \{x_3\}$, and we have $\int_{\omega_s} x_{s1}\, d\omega_s = \int_{\omega_s} x_{s2}\, d\omega_s = \int_{\omega_s} x_{s1}x_{s2}\, d\omega_s = 0$. (We observe that the choice of the vector field θ, which realizes the shape variation of the cross section ω, must be admissible with this condition.)

The set $C^m(\overline{\Omega}_s)$ is the space of real functions m times continuously differentiable in $\overline{\Omega}_s$. The spaces $W^{m,q}(\Omega_s)$ and $W^{0,q}(\Omega_s) = L^q(\Omega_s)$ are the usual Sobolev spaces, where q is a real number satisfying $1 \le q \le \infty$ and m is a positive integer. The norms in these Sobolev spaces are denoted by $\|.\|_{W^{m,q}(\Omega_s)}$. The set $\mathcal{R}_s = \{v_s \in \mathbb{R}^3 : \ v_s = a + b \wedge x_s,\ a, b \in \mathbb{R}^3\}$, where $\wedge$ is the exterior product in $\mathbb{R}^3$, is the set of infinitesimal rigid displacements. We denote by $[W^{m,q}(\Omega_s)]^3/\mathcal{R}_s$ the quotient space induced by the set $\mathcal{R}_s$ in the Sobolev space $[W^{m,q}(\Omega_s)]^3$.

Throughout the paper, the Latin indices i, j, k, $l, \ldots$ belong to the set $\{1,2,3\}$; the Greek indices α, β, $\mu, \ldots$ vary in the set $\{1,2\}$; and the summation convention with respect to repeated indices is employed, that is, for example, $a_ib_i = \sum_{i=1}^3 a_ib_i$.

Let $T > 0$ be a real parameter, and denote by t the time variable in the interval $[0,T]$. If V is a topological vectorial space, the set $C^m([0,T];V)$ is the space of functions $g : t \in [0,T] \to g(t) \in V$ such that g is m times continuously differentiable with respect to t. If V is a Banach space, we denote by $\|.\|_{C^m([0,T];V)}$ the usual norm in $C^m([0,T];V)$. Moreover, given a function $g_s(x_s,t)$ defined in $\overline{\Omega}_s \times [0,T]$, we denote by $\dot{g}_s$ its partial derivative with respect to time and by $\partial_{s\alpha} g_s$ and $\partial_3 g_s$ its partial derivatives with respect to $x_{s\alpha}$ and x_3; that is, $\dot{g}_s = \frac{\partial g_s}{\partial t}$, $\partial_{s\alpha} g_s = \frac{\partial g_s}{\partial x_{s\alpha}}$, and $\partial_3 g_s = \frac{\partial g_s}{\partial x_3}$.

2.2. The adaptive elastic rod model. Figueiredo and Trabucho [7] have applied the asymptotic expansion method to the three-dimensional adaptive elasticity model derived by Cowin and Hegedus [3, 9], with the modifications proposed by Monnier and Trabucho [10], for a thin rod whose cross section is a function of a small parameter and for a three-dimensional remodeling rate equation depending nonlinearly or linearly on the strain tensor field (cf. also Trabucho and Viaño [11] for an explanation of the mathematical modeling of rods with the asymptotic expansion method). They have obtained a simplified adaptive elastic rod model, which is designated in what follows by the asymptotic adaptive elastic rod model. This is a system of nonlinear coupled equations, which includes generalized Bernoulli–Navier equilibrium equations and a simplified remodeling rate equation. For any perturbed rod $\overline{\Omega}_s$ (with $s \in [0,\delta]$) and for the case where the original three-dimensional remodeling rate equation depends linearly on the strain tensor field, this system is defined as follows:

$$
(2.1)\qquad \left[\begin{array}{l}
u_s = (u_{s1}, u_{s2}, u_{s3}) : \overline{\Omega}_s \times [0,T] \to \mathbb{R}^3, \quad d_s : \overline{\Omega}_s \times [0,T] \to \mathbb{R},\\
u_{s\alpha} : [0,L] \times [0,T] \to \mathbb{R},\\
u_{s3} = \underline{u}_{s3} - x_{s\alpha}\partial_3 u_{s\alpha} \quad \text{and} \quad \underline{u}_{s3} : [0,L] \times [0,T] \to \mathbb{R},\\
(u_s, d_s) \quad \text{satisfies the following:}
\end{array}\right.
$$

$$
(2.2)\quad \left[\begin{array}{l}
\textit{Equilibrium equations in } (0,L)\times(0,T)\\
\left[\begin{array}{l} -\partial_3\big(l_s(d_s)\partial_3\underline{u}_{s3} - e_{s\alpha}(d_s)\partial_{33}u_{s\alpha}\big)\\
\quad = \int_{\omega_s}\gamma_s(\xi_{s0}+P_\eta(d_s))\,f_{s3}\,d\omega_s + \int_{\partial\omega_s} g_{s3}\,d\partial\omega_s,\end{array}\right.\\
\left[\begin{array}{l} \partial_{33}\big(-e_{s\beta}(d_s)\partial_3\underline{u}_{s3} + h_{s\alpha\beta}(d_s)\partial_{33}u_{s\alpha}\big)\\
= \int_{\omega_s}\gamma_s(\xi_{s0}+P_\eta(d_s))\,f_{s\beta}\,d\omega_s + \int_{\partial\omega_s} g_{s\beta}\,d\partial\omega_s\\
+ \int_{\omega_s} x_{s\beta}\,\partial_3[\gamma_s(\xi_{s0}+P_\eta(d_s))\,f_{s3}]\,d\omega_s + \int_{\partial\omega_s} x_{s\beta}\,\partial_3 g_{s3}\,d\partial\omega_s,\qquad \beta=1,2,\end{array}\right.
\end{array}\right.
$$

$$
(2.3)\quad \left[\begin{array}{l}
\textit{Boundary conditions for } \{\overline{x}_3\}\times(0,T), \text{ with } \overline{x}_3 = 0, L\\
\big(l_s(d_s)\partial_3\underline{u}_{s3} - e_{s\alpha}(d_s)\partial_{33}u_{s\alpha}\big)(\overline{x}_3) = \int_{\omega_s} h_{s3}(\overline{x}_3)\,d\omega_s,\\
\big(e_{s\beta}(d_s)\partial_3\underline{u}_{s3} - h_{s\alpha\beta}(d_s)\partial_{33}u_{s\alpha}\big)(\overline{x}_3) = \int_{\omega_s} x_{s\beta}\,h_{s3}(\overline{x}_3)\,d\omega_s,\\
\left[\begin{array}{l} \partial_3\big(e_{s\beta}(d_s)\partial_3\underline{u}_{s3} - h_{s\alpha\beta}(d_s)\partial_{33}u_{s\alpha}\big)(\overline{x}_3)\\
\quad = \int_{\omega_s} h_{s\beta}(\overline{x}_3)\,d\omega_s - \int_{\partial\omega_s} x_{s\beta}g_{s3}(\overline{x}_3)\,d\partial\omega_s\\
\qquad - \int_{\omega_s} x_{s\beta}\,\gamma_s(\xi_{s0}+P_\eta(d_s))f_{s3}(\overline{x}_3)\,d\omega_s,\end{array}\right.
\end{array}\right.
$$

$$
(2.4)\quad \left[\begin{array}{lll}
\textit{Remodeling rate equation} & & \\
\dot{d}_s = c(d_s)e_{33}(u_s) + a(d_s) & \text{in} & \Omega_s\times(0,T),\\
c(d_s) = A_{\alpha\beta}(d_s)\frac{b_{\alpha\beta33}(d_s)}{b_{3333}(d_s)} + A_{33}(d_s), & & \\
d_s(x_s,0) = \hat{d}_s(x_s) & \text{in} & \overline{\Omega}_s.
\end{array}\right.
$$

The unknowns of the model (2.1)–(2.4) are the displacement vector field $u_s(x_s,t)$, corresponding to the displacement of the point x_s of the rod $\overline{\Omega}_s$ at time t, and the measure of change in volume fraction of the elastic material (from the reference volume fraction ξ_{s0}) $d_s(x_s,t)$ at (x_s,t). In particular, $e_{33}(u_s) = \partial_3 u_{s3} = \partial_3\underline{u}_{s3} - x_{s\alpha}\partial_{33}u_{s\alpha}$ is an element of the linear strain tensor field $(e_{ij}(u_s))$, which depends on u_s.

On the other hand, the data of the model (2.1)–(2.4) are the following: the open set $\Omega_s\times(0,T)$; the density $\gamma_s=\gamma$ of the full elastic material, which is supposed to be a constant independent of s; the reference volume fraction of the elastic material ξ_{s0}, which belongs to $C^1(\overline{\Omega}_s)$ for each s; the body load $f_s=(f_{si})$ such that $f_{si}\in C^1([0,T])$ and depends only on t; the normal tractions on the boundary $g_s=(g_{si})$ and $h_s=(h_{si})$; the initial value of the change in volume fraction $\hat{d}_s$, which belongs to $C^0(\overline{\Omega}_s)$; the truncation operator $\mathcal{P}_\eta(.)$; and the coefficients $l_s(d_s)$, $e_{s\alpha}(d_s)$, $h_{s\alpha\beta}(d_s)$, $c(d_s)$, $a(d_s)$, $A_{\alpha\beta}(d_s)$, $A_{33}(d_s)$, $b_{\alpha\beta33}(d_s)$, and $b_{3333}(d_s)$, which are all material coefficients depending upon the change in volume fraction d_s.

On these data we also suppose further conditions, which we will describe next. We assume that, for each $s\in[0,\delta]$, $0<\xi_{s0}^{min}\le\xi_{s0}(x_s)\le\xi_{s0}^{max}<1$ and the normal tractions verify

$$
(2.5)\qquad g_{si}\in C^1([0,T];W^{1-1/p,p}(\Gamma_s)),\quad h_{si}\in C^1([0,T];W^{1-1/p,p}(\Gamma_{s0}\cup\Gamma_{sL})),
$$

with $p>3$. In addition, we assume that the resultant of the system of applied forces is null for rigid displacements; this means that, for any $v_s=(v_{si})$ in $\mathcal{R}_s$ and for all $t\in[0,T]$,

$$
(2.6)\qquad \int_{\Omega_s}\gamma(\xi_{s0}+\mathcal{P}_\eta(d_s))f_{si}v_{si}dx_s + \int_{\Gamma_s} g_{si}v_{si}d\Gamma_s + \int_{\Gamma_{s0,L}} h_{si}v_{si}d\Gamma_{s0,L} = 0.
$$

The truncation operator $\mathcal{P}_\eta$ is of class C^1 and satisfies $0 < \frac{\eta}{2} \le (\xi_{s0} + \mathcal{P}_\eta(d_s))(x_s) \le 1$ for all $x_s \in \overline{\Omega}_s$, where $\eta > 0$ is a small parameter.

The coefficients $b_{\alpha\beta33}(d_s)$ and $b_{3333}(d_s)$ are continuously differentiable with respect to d_s and are elements of the matrix $(b_{ijkl}(d_s))$, which is the inverse of the matrix defined by the three-dimensional elastic coefficients $(c_{ijkl}(d_s))$ of the rod $\overline{\Omega}_s$, that depend on d_s through truncation and mollification (cf. formulas (47)–(48), Figueiredo and Trabucho [7]). The coefficients $A_{\alpha\beta}(d_s)$, $A_{33}(d_s)$, $c(d_s)$, and $a(d_s)$ are remodeling rate coefficients and are continuously differentiable with respect to d_s.

Moreover, $b_{\alpha\beta33}(d_s)$ and $b_{3333}(d_s)$ belong to the space $C^1([0,T]; C^1(\mathbb{R}^3))$ when $d_s \in C^1([0,T]; C^0(\overline{\Omega}_s))$ (cf. Monnier and Trabucho [10, p. 542] and also formulas (47)–(48) of Figueiredo and Trabucho [7]). In addition we also assume that there exist strictly positive constants C_1, C_2, C_3, C_4, C_5, and C_6 independent of s and t such that for any $(x_s, t) \in \overline{\Omega}_s \times [0,T]$

$$0 < C_1 \le \frac{1}{b_{3333}(d_s)} \le C_2 \quad \forall s \in [0,\delta], \tag{2.7}$$

$$|c(d_s)| \le C_3,\ |a(d_s)| \le C_4,\ |c'(d_s)| \le C_5,\ |a'(d_s)| \le C_6 \quad \forall s \in [0,\delta], \tag{2.8}$$

where $c'(.)$ and $a'(.)$ are the derivatives of the scalar functions $c(.)$ and $a(.)$. We remark that the assumption (2.7) is a direct consequence of the definition of b_{3333}, and also a consequence of [10, Lemma 1, p. 542]. The assumption (2.8) can be proven using exactly the same arguments of this Lemma 1, supposing that $c(d_s)$ and $a(d_s)$ depend on d_s through truncation and mollification.

The coefficients $l_s(d_s)$, $e_{s\alpha}(d_s)$, and $h_{s\alpha\beta}(d_s)$, which depend on $b_{3333}(d_s)$ (cf. formula (49), Figueiredo and Trabucho [7]), are functions of x_3 and t, and are defined by

$$l_s = \int_{\omega_s} \frac{1}{b_{3333}(d_s)} d\omega_s, \qquad e_{s\alpha} = \int_{\omega_s} \frac{x_{s\alpha}}{b_{3333}(d_s)} d\omega_s, \qquad h_{s\alpha\beta} = \int_{\omega_s} \frac{x_{s\alpha} x_{s\beta}}{b_{3333}(d_s)} d\omega_s. \tag{2.9}$$

The variational formulation of the equilibrium equations (2.2) is obtained by multiplying the first equilibrium equation (2.2) by $\underline{v}_{s3} \in W^{1,2}(0,L)$ and the second and third equilibrium equations by $x_{s1}\, v_{s1}$ and $x_{s2}\, v_{s2}$, respectively, with $v_{s\beta} \in W^{2,2}(0,L)$, for $\beta = 1, 2$, and subsequently integrating in $(0,L)$ and using the boundary conditions (2.3). Thus, the asymptotic adaptive elastic rod model (2.1)–(2.4) is equivalent to the following nonlinear (variational and differential) system (P_s) (cf. formula (56) of Figueiredo and Trabucho [7]):

$$(\mathrm{P}_s) \qquad \left[\begin{array}{l} \text{Find} \quad u_s : \overline{\Omega}_s \times [0,T] \to \mathbb{R}^3 \quad \text{and} \quad d_s : \overline{\Omega}_s \times [0,T] \to \mathbb{R} : \\ u_s(.,t) \in V(\Omega_s)/\mathcal{R}_s, \\ a_s(u_s, v_s) = L_s(v_s) \qquad \forall v_s \in V(\Omega_s)/\mathcal{R}_s, \\ \dot{d}_s = c(d_s) e_{33}(u_s) + a(d_s) \quad \text{in} \quad \Omega_s \times (0,T), \\ d_s(x_s, 0) = \hat{d}_s(x_s) \quad \text{in} \quad \overline{\Omega}_s. \end{array}\right.$$

The space $V(\Omega_s) = \{v_s \in [W^{1,2}(\Omega_s)]^3 : \ e_{\alpha\beta}(v_s) = e_{3\beta}(v_s) = 0\}$ is identified with

$$\begin{aligned} \Big\{ & v_s = (v_{s1}, v_{s2}, v_{s3}) \in [W^{2,2}(0,L)]^2 \times W^{1,2}(\Omega_s) : \quad v_{s\alpha}(x_s) = v_{s\alpha}(x_3), \\ & v_{s3}(x_s) = \underline{v}_{s3}(x_3) - x_{s\alpha} \partial_3 v_{s\alpha}(x_3), \quad \underline{v}_{s3} \in W^{1,2}(0,L) \Big\}, \end{aligned} \tag{2.10}$$

and the quotient space $V(\Omega_s)/\mathcal{R}_s$ is the following set:

$$\Big\{v_s = z_s + a + b \wedge x_s : \quad z_s \in V(\Omega_s),\ a \in \mathbb{R}^3, \quad b = (b_1, b_2, 0) \in \mathbb{R}^3\Big\}. \tag{2.11}$$

The bilinear form $a_s(.,.)$, depending on the unknown d_s, is defined in $V(\Omega_s)/\mathcal{R}_s$ by

$$a_s(z_s, v_s) = \int_{\Omega_s} \frac{1}{b_{3333}(d_s)} e_{33}(z_s) e_{33}(v_s) d\Omega_s \quad \forall z_s,\ v_s \in V(\Omega_s)/\mathcal{R}_s, \tag{2.12}$$

and $L_s(.)$ is a linear form also defined in $V(\Omega_s)/\mathcal{R}_s$ such that $L_s(v_s)$ is equal to

$$\int_{\Omega_s} \gamma(\xi_{s0} + P_\eta(d_s))\, f_{si}\, v_{si}\, d\Omega_s + \int_{\Gamma_s} g_{si}\, v_{si}\, d\Gamma_s + \int_{\Gamma_{s0,L}} h_{si}\, v_{si}\, d\Gamma_{s0,L}. \tag{2.13}$$

We remark that in (2.11) we must have $b_3 = 0$, because otherwise the quotient space $V(\Omega_s)/\mathcal{R}_s$ would not be contained in $V(\Omega_s)$. In fact, developing $v_s = z_s + a + b \wedge x_s$, we have for the first component $v_{s1} = z_{s1} + a_1 + b_2 x_3 - b_3 x_{s2}$, for the second component $v_{s2} = z_{s2} + a_2 - b_1 x_3 + b_3 x_{s1}$, and finally for the third component $v_{s3} = \underline{z}_{s3} - x_{s\alpha}\partial_3 z_{s\alpha} + a_3 + b_1 x_{s2} - b_2 x_{s1}$. Therefore if $b_3 \neq 0$, then $v_s \notin V(\Omega_s)$, and if $b_3 = 0$, then $(b_1, b_2, 0) \wedge x_s = \big(b_2 x_3, -b_1 x_3, b_1 x_{s2} - b_2 x_{s1}\big)$ and we obtain $v_{s1} = z_{s1} + a_1 + b_2 x_3$, which depends only on x_3, $v_{s2} = z_{s2} + a_2 - b_1 x_3$, which depends only on x_3, and $v_{s3} = \underline{v}_{s3} - x_{s\alpha}\partial_3 v_{s\alpha}$ with $\underline{v}_{s3} = \underline{z}_{s3} + a_3$, which depends only on x_3.

By the following Korn-type inequality in the quotient space $V(\Omega_s)/\mathcal{R}_s$ (cf. Ciarlet [1] or Valent [12]) we have

$$\exists c > 0 : \quad \|v_s\|^2_{[W^{1,2}(\Omega_s)]^3} \leq c\|e_{33}(v_s)\|^2_{L^2(\Omega_s)}, \tag{2.14}$$

where

$$\|e_{33}(v_s)\|^2_{L^2(\Omega_s)} = c_s\|\partial_3\underline{v}_{s3}\|^2_{L^2(0,L)} + \left(\int_{\omega_s} x^2_{s\alpha} d\omega_s\right)\|\partial_{33}v_{s\alpha}\|^2_{L^2(0,L)}, \tag{2.15}$$

with $c_s = [\mathrm{meas}(\omega_s)]^{\frac{1}{2}}$, since $e_{33}(v_s) = \partial_3 v_{s3} = \partial_3\underline{v}_{s3} - x_{s\alpha}\partial_{33}v_{s\alpha}$. Hence, we conclude that $\|e_{33}(.)\|_{L^2(\Omega_s)}$ is a norm in the space $V(\Omega_s)/\mathcal{R}_s$, equivalent to the usual norm induced in the quotient space by $\|.\|_{[W^{1,2}(\Omega_s)]^3}$. Moreover, $V(\Omega_s)/\mathcal{R}_s$ is a Hilbert space with the norm $\|e_{33}(.)\|_{L^2(\Omega_s)}$, and the bilinear form $a_s(.,.)$ is elliptic in $V(\Omega_s)/\mathcal{R}_s$. In fact, there exists a constant $C > 0$ such that

$$\left\{\begin{array}{l} a_s(v_s, v_s) = \int_{\Omega_s} \frac{1}{b_{3333}(d_s)}\, e_{33}(v_s)\, e_{33}(v_s)\, d\Omega_s \geq C_1\|e_{33}(v_s)\|^2_{L^2(\Omega_s)} \\ = C_1\|v_s\|^2_{V(\Omega_s)/\mathcal{R}_s} \geq C\|v_s\|^2_{[W^{1,2}(\Omega_s)]^3} \quad \forall v_s \in V(\Omega_s)/\mathcal{R}_s, \end{array}\right. \tag{2.16}$$

where C_1 is the constant defined in condition (2.7).

For each $s \in [0, \delta]$, there exists a unique pair (u_s, d_s) solution of the asymptotic adaptive elastic rod model P_s, which verifies $u_s \in C^1([0,T]; V(\Omega_s)/\mathcal{R}_s)$ and $d_s \in C^1([0,T]; C^0(\overline{\Omega}_s))$ (cf. Theorem 6, Figueiredo and Trabucho [7]). The next theorem states a regularity result, concerning the component solution u_s, that will be important in section 4. In order to prove it, we introduce the following notation:

$$\begin{aligned} & z_{s3} = l_s\partial_3\underline{u}_{s3} - e_{s\alpha}\partial_{33}u_{s\alpha}, \quad z_{s\beta} = h_{s\alpha\beta}\partial_{33}u_{s\alpha} - e_{s\beta}\partial_3\underline{u}_{s3}, \\ & F_{s3} = \int_{\omega_s} \gamma(\xi_{s0} + P_\eta(d_s))\, f_{s3}\, d\omega_s + \int_{\partial\omega_s} g_{s3}\, d\partial\omega_s, \\ & F_{s\beta} = \left\{\begin{array}{l} \int_{\omega_s} \gamma(\xi_{s0} + P_\eta(d_s))\, f_{s\beta}\, d\omega_s + \int_{\partial\omega_s} g_{s\beta}\, d\partial\omega_s \\ \quad + \int_{\omega_s} x_{s\beta}\, \partial_3[\gamma(\xi_{s0} + P_\eta(d_s))\, f_{s3}]\, d\omega_s + \int_{\partial\omega_s} x_{s\beta}\, \partial_3 g_{s3}\, d\partial\omega_s, \end{array}\right. \end{aligned} \tag{2.17}$$

where $z_{si} \in C^1([0,T];L^2(0,L))$, $F_{si} \in C^1([0,T];L^2(0,L))$, for $i = 1,2,3$, and the matrix M_s,

$$M_s = \begin{bmatrix} l_s & -e_{s1} & -e_{s2} \\ -e_{s1} & h_{s11} & h_{s12} \\ -e_{s2} & h_{s21} & h_{s22} \end{bmatrix} \in C^1\big([0,T];[C^1(\mathbb{R}^3)]^9\big). \tag{2.18}$$

THEOREM 2.1 (regularity of u_s). *Let (u_s, d_s) be the unique solution of problem* (P$_s$). *We assume that the determinant of matrix M_s is not zero,* $\det M_s \neq 0$ *(for example, if $b_{3333}(d^s) = c$, where c is a constant, then* $\det M_s > 0$*). Then, for each $t \in [0,T]$, $u_{s\beta}(.,t) \in W^{3,2}(0,L)$, $\underline{u}_{s3}(.,t) \in W^{2,2}(0,L)$, and consequently $u_s(.,t) \in [W^{2,2}(\Omega_s)]^3$.*

Proof. We first remark that the equilibrium equations (2.2) can be written in the form

$$-\partial_3\, z_{s3} = F_{s3}, \qquad \partial_{33}\, z_{s\beta} = F_{s\beta}, \quad \text{for } \beta = 1,2, \tag{2.19}$$

and for each t, $F_{s3}(.,t)$ and $F_{s\beta}(.,t)$ belong to the space $L^2(0,L)$.

Since $z_{s3}(.,t)$ belongs to $L^2(0,L)$, and because of the first equilibrium equation in (2.19), $\partial_3\, z_{s3}(.,t)$ also belongs to $L^2(0,L)$, so we conclude that, for each t, $z_{s3}(.,t) \in W^{1,2}(0,L)$. For each t, $z_{s\beta}(.,t) \in L^2(0,L)$ and consequently $\partial_3\, z_{s\beta}(.,t) \in [W^{1,2}(0,L)]'$, where $[W^{1,2}(0,L)]'$ is the dual of $W^{1,2}(0,L)$. But from the second equilibrium equation in (2.19) we have that $\partial_{33}\, z_{s\beta}(.,t) \in L^2(0,L)$ and also $\partial_{33}\, z_{s\beta}(.,t) \in [W^{1,2}(0,L)]'$ because $L^2(0,L) \subset [W^{1,2}(0,L)]'$. Thus, as a consequence of a lemma of Lions (cf. Ciarlet [2, p. 39]) we have $\partial_3\, z_{s\beta}(.,t) \in L^2(0,L)$. Hence the elements $z_{s\beta}(.,t)$, $\partial_3\, z_{s\beta}(.,t)$, and $\partial_{33}\, z_{s\beta}(.,t)$ belong to the space $L^2(0,L)$, which means that $z_{s\beta}(.,t) \in W^{2,2}(0,L)$.

Therefore, assembling these properties, we obtain for each t and for $\beta = 1,2$

$$z_{s3}(.,t) = p_{s3}(.,t) \in W^{1,2}(0,L), \qquad z_{s\beta}(.,t) = p_{s\beta}(.,t) \in W^{2,2}(0,L), \tag{2.20}$$

where p_{si} is a primitive of F_{si}, in the distribution's sense in $W^{1,2}(0,L)$, for $i = 1,2,3$. Replacing z_{si} by its definition (2.17), the system (2.20) is equivalent to the following system:

$$\begin{bmatrix} l_s & -e_{s1} & -e_{s2} \\ -e_{s1} & h_{s11} & h_{s12} \\ -e_{s2} & h_{s21} & h_{s22} \end{bmatrix} \begin{bmatrix} \partial_3 \underline{u}_{s3} \\ \partial_{33} u_{s1} \\ \partial_{33} u_{s2} \end{bmatrix} = \begin{bmatrix} p_{s3} \\ p_{s1} \\ p_{s2} \end{bmatrix}. \tag{2.21}$$

With the assumption $\det M_s \neq 0$, we clearly obtain, by solving (2.21), that

$$\partial_3 \underline{u}_{s3}(.,t) \in W^{1,2}(0,L), \qquad \partial_{33} u_{s\beta}(.,t) \in W^{1,2}(0,L), \quad \text{for } \beta = 1,2. \tag{2.22}$$

We remark that the regularity indicated in (2.22) depends also on the regularity of the elements of M_s that belong to the space $C^1([0,T];C^1(\mathbb{R}^3))$.

Thus we conclude that the components $u_{s\beta}(.,t) \in W^{3,2}(0,L)$, for $\beta = 1,2$ and $\underline{u}_{s3}(.,t) \in W^{2,2}(0,L)$, and consequently, because $u_s = (u_{s1}, u_{s2}, \underline{u}_{s3} - x_{s\beta}\partial_3\, u_{s\beta})$, we have $u_s(.,t) \in [W^{2,2}(\Omega_s)]^3$. $\square$

2.3. The shape problem. We now consider the shape map J defined by

$$\begin{array}{llll} J: & [0,\delta] & \longrightarrow & C^1([0,T];V(\Omega_s)/\mathcal{R}_s) \times C^1([0,T];C^0(\overline{\Omega}_s)) \\ & s & \longrightarrow & J(\Omega_s) = (u_s, d_s), \end{array} \tag{2.23}$$

where (u_s, d_s) is the unique solution of the nonlinear asymptotic adaptive rod model P_s (cf. (P_s)), defined in the perturbed rod Ω_s. As remarked before, the unknowns u_s and d_s are coupled in the model P_s and depend on (x_s, t).

We recall that $I_s(\omega)$ is a shape perturbation of the cross section ω of the rod $\overline{\Omega} = \overline{\omega} \times [0, L]$, so $\overline{\Omega}_s$ is a shape perturbation of the rod $\overline{\Omega}$; that is, $\overline{\Omega}_s = I_s(\overline{\omega}) \times [0, L] = (I + s\,\theta)(\overline{\omega}) \times [0, L]$ and $\overline{\Omega} = \overline{\omega} \times [0, L] = I_0(\overline{\omega}) \times [0, L] = \overline{\Omega}_0$.

The aim of this paper is to compute the shape semiderivative $dJ(\Omega; \theta)$ at $s = 0$ in the direction of the vector field θ. This semiderivative is defined by (cf. Delfour and Zolésio [5, p. 289])

$$dJ(\Omega; \theta) = \lim_{s \to 0^+} \frac{J(\Omega_s) - J(\Omega)}{s} = \left(\lim_{s \to 0^+} \frac{u_s - u}{s}, \ \lim_{s \to 0^+} \frac{d_s - d}{s} \right), \tag{2.24}$$

where (u, d) is the solution of problem (P_s) but for the unperturbed rod $\overline{\Omega}_0 = \overline{\Omega} = \overline{\omega} \times [0, L]$. We also remark that the semiderivative $dJ(\Omega; \theta)$ is equivalent to the definition of the material derivative of the map J at $s = 0$, in the sense of Haslinger and Mäkinen [8, p. 111].

As explained in section 4, Theorem 4.11, it is possible to compute and to identify, in a weak sense and in an appropriate product space, this shape semiderivative.

3. Equivalent formulation of the adaptive rod model. In order to be able to calculate the shape semiderivative (2.24) we must reformulate, for each s, the problem P_s (cf. (P_s)) in the domain $\overline{\Omega} \times [0, T]$, independent of s. Therefore we first formulate in Ω all the forms involved in the definition of problem P_s. Afterwards, we describe the resulting rod model, denoted by P^s (with upper index s) and formulated in the fixed domain $\overline{\Omega} \times [0, T]$, that is equivalent to P_s (with lower index s).

3.1. Reformulation of the forms defining $\mathbf{P_s}$. We define the map

$$Q_s(x_1, x_2, x_3) = \big(x_1 + s\theta_1(x_1, x_2), x_2 + s\theta_2(x_1, x_2), x_3\big), \tag{3.1}$$

which verifies

$$\Omega_s = Q_s(\Omega) \quad \text{and} \quad \det \nabla Q_s = 1 + s \operatorname{div} \theta + s^2 \det \nabla \theta, \tag{3.2}$$

where the matrices ∇Q_s and $\nabla \theta$ are the gradients of Q_s and θ, respectively, and $\operatorname{div} \theta = \partial_\alpha \theta_\alpha$ is the divergence of θ.

To each function v_s defined in Ω_s we associate the corresponding function v^s (with upper index s) defined on Ω by $v^s = v_s \circ Q_s$. Hence, for any $v_s \in V(\Omega_s)/\mathcal{R}_s$, the correspondent v^s is in $V(\Omega)/\mathcal{R}$ (whose definition is (2.11), with $s = 0$). Moreover,

$$\begin{aligned} e_{33}(v_s) &= \partial_3 \underline{v}_{s3} - x_{s\alpha} \partial_{33} v_{s\alpha} = \partial_3 \underline{v}_{s3} - (x_\alpha + s\,\theta_\alpha) \partial_{33} v^s_\alpha \\ &= \partial_3 \underline{v}^s_3 - x_\alpha \partial_{33} v^s_\alpha - s\,\theta_\alpha \partial_{33} v^s_\alpha = e_{33}(v^s) - s\,\theta_\alpha \partial_{33} v^s_\alpha. \end{aligned} \tag{3.3}$$

Using (3.2)–(3.3) and the change of variables formula for domain and boundary integrals (cf. Delfour and Zolésio [5, pp. 351–353]), we get the next expression for $a_s(u_s, v_s)$,

$$\int_\Omega \frac{1}{b_{3333}(d^s)} \big(e_{33}(u^s) - s\,\theta_\alpha \partial_{33} u^s_\alpha\big)\big(e_{33}(v^s) - s\,\theta_\alpha \partial_{33} v^s_\alpha\big) \det \nabla Q_s d\Omega, \tag{3.4}$$

and for $L_s(v_s)$ the expression

$$\text{(3.5)}\quad \left\{ \begin{array}{l} \int_\Omega \gamma(\xi_0^s + P_\eta(d^s))\, f_i^s\, v_i^s\, (\det \nabla Q_s)\, d\Omega + \int_\Gamma g_i^s\, v_i^s\, |(Cof\, \nabla Q_s)^T n|_{\mathbb{R}^3}\, d\Gamma \\ \qquad + \int_{\Gamma_0 \cup \Gamma_L} h_i^s\, v_i^s\, |(Cof\, \nabla Q_s)^T n|_{\mathbb{R}^3}\, d(\Gamma_0 \cup \Gamma_L). \end{array} \right.$$

In (3.5), $|.|_{\mathbb{R}^3}$ is the Euclidean norm in $\mathbb{R}^3$, $n = (n_1, n_2, n_3)$ is the unit outer normal vector along the boundary $\partial\Omega$ of Ω, and $(Cof\, \nabla Q_s)^T$ is the transpose of the cofactor matrix of ∇Q_s, that is, $(Cof\, \nabla Q_s)^T = (\det \nabla Q_s)(\nabla Q_s)^{-T}$, whose definition depends only on s and the partial derivatives of θ. Developing (3.4)–(3.5), we obtain the following decomposition for the equation $a_s(u_s, v_s) = L_s(v_s)$:

$$\text{(3.6)}\quad \left[\begin{array}{l} a_0^s(u^s,v^s) + s a_1^s(u^s,v^s) + s^2 a_2^s(u^s,v^s) + s^3 a_3^s(u^s,v^s) + s^4 a_4^s(u^s,v^s) \\ = \left\{ \begin{array}{l} F_0^s(v^s) + G_0^s(v^s) + H_0^s(v^s) + s\left(F_1^s(v^s) + G_1^s(v^s) + H_1^s(v^s)\right) \\ + s^2\left(F_2^s(v^s) + G_2^s(v^s) + H_2^s(v^s)\right) + s^3\left(F_3^s(v^s) + G_3^s(v^s) + H_3^s(v^s)\right). \end{array} \right. \end{array} \right.$$

The bilinear forms $a_i^s(.,.)$ for $i = 0,1,2,3,4$ depend on θ and d^s and are defined by the formulas

$$\begin{aligned}
a_0^s(u,v) &= \int_\Omega \frac{1}{b_{3333}(d^s)} e_{33}(u) e_{33}(v) d\Omega, \\
a_1^s(u,v) &= \left\{ \int_\Omega \frac{1}{b_{3333}}(d^s)\Big[-\theta_\alpha\big(e_{33}(u)\partial_{33}v_\alpha + e_{33}(v)\partial_{33}u_\alpha\big) + e_{33}(u)e_{33}(v)\operatorname{div}\theta\Big] d\Omega, \right. \\
a_2^s(u,v) &= \left\{ \int_\Omega \frac{1}{b_{3333}(d^s)}\Big[e_{33}(u)e_{33}(v)\det\nabla\theta + \theta_\alpha\,\theta_\beta\,\partial_{33}u_\alpha\,\partial_{33}v_\beta - (\operatorname{div}\theta)\,\theta_\alpha\big(e_{33}(u)\partial_{33}v_\alpha + e_{33}(v)\partial_{33}u_\alpha\big)\Big] d\Omega, \right. \quad \text{(3.7)} \\
a_3^s(u,v) &= \left\{ \int_\Omega \frac{1}{b_{3333}}(d^s)\Big[\theta_\alpha\theta_\beta\partial_{33}u_\alpha\partial_{33}v_\beta\operatorname{div}\theta - (\det\nabla\theta)\,\theta_\alpha\big(e_{33}(u)\partial_{33}v_\alpha + e_{33}(v)\partial_{33}u_\alpha\big)\Big] d\Omega, \right. \\
a_4^s(u,v) &= \int_\Omega \frac{1}{b_{3333}(d^s)}\Big[\theta_\alpha\,\theta_\beta\,\partial_{33}u_\alpha\,\partial_{33}v_\beta\,\det\nabla\theta\Big] d\Omega
\end{aligned}$$

for any pair (u,v) in the space $V(\Omega)/\mathcal{R}$. The linear forms F_0^s, F_1^s, F_2^s, and F_3^s depend on θ and d^s and are also defined in the same quotient space $V(\Omega)/\mathcal{R}$ by the following expressions:

$$\text{(3.8)}\quad \begin{aligned}
F_0^s(v) &= \int_\Omega \gamma(\xi_0^s + P_\eta(d^s))\,(f_\alpha^s\,v_\alpha + f_3^s\,\underline{v}_3)\,d\Omega, \\
F_1^s(v) &= \int_\Omega \gamma(\xi_0^s + P_\eta(d^s))\,\Big[(f_\alpha^s\,v_\alpha + f_3^s\,\underline{v}_3)\operatorname{div}\theta - f_3^s\,\theta_\alpha\,\partial_3 v_\alpha\Big]\,d\Omega, \\
F_2^s(v) &= \int_\Omega \gamma(\xi_0^s + P_\eta(d^s))\,\Big[(f_\alpha^s\,v_\alpha + f_3^s\,\underline{v}_3)\,\det\nabla\theta - f_3^s\,\theta_\alpha\,\partial_3 v_\alpha\operatorname{div}\theta\Big]\,d\Omega, \\
F_3^s(v) &= -\int_\Omega \gamma(\xi_0^s + P_\eta(d^s))\,f_3^s\,\theta_\alpha\,\partial_3 v_\alpha\,\det\nabla\theta\,d\Omega.
\end{aligned}$$

The linear forms G_0^s, G_1^s, G_2^s, G_3^s and H_0^s, H_1^s, H_2^s, H_3^s result from the change of variable in the boundary integrals (defined in Γ_s and in $\Gamma_{s0} \cup \Gamma_{sL}$, respectively) and depend on θ and n (the unit outer normal vector) but are independent of d^s. These forms are defined by the following expressions, for any v in the space $V(\Omega)/\mathcal{R}$:

$$
\begin{aligned}
G_0^s(v) &= \int_\Gamma \big(g_\alpha^s\, v_\alpha + g_3^s\, \underline{v}_3\big)\, d\Gamma,\\
G_1^s(v) &= \int_\Gamma \Big[\big(g_\alpha^s\, v_\alpha + g_3^s\, \underline{v}_3\big)\, G_1(\theta,n) - g_3^s\, \theta_\alpha\, \partial_3 v_\alpha\Big]\, d\Gamma,\\
G_2^s(v) &= \int_\Gamma \Big[\big(g_\alpha^s\, v_\alpha + g_3^s\, \underline{v}_3\big)\, G_2(\theta,n) - g_3^s\, \theta_\alpha\, \partial_3 v_\alpha\, G_1(\theta,n)\Big]\, d\Gamma,\\
G_3^s(v) &= -\int_\Gamma g_3^s\, \theta_\alpha\, \partial_3 v_\alpha\, G_3(\theta,n)\, d\Gamma,
\end{aligned}
\tag{3.9}
$$

where $G_1(\theta,n)$, $G_2(\theta,n)$, and $G_3(\theta,n)$ are bounded scalar functions of θ and n and

$$
\begin{aligned}
H_0^s(v) &= \int_{\Gamma_0\cup\Gamma_L} \big(h_\alpha^s\, v_\alpha + h_3^s\, \underline{v}_3\big)\, d(\Gamma_0\cup\Gamma_L),\\
H_1^s(v) &= \int_{\Gamma_0\cup\Gamma_L} \Big[\big(h_\alpha^s\, v_\alpha + h_3^s\, \underline{v}_3\big)\, H_1(\theta) - h_3^s\, \theta_\alpha\, \partial_3 v_\alpha\Big]\, d(\Gamma_0\cup\Gamma_L),\\
H_2^s(v) &= \int_{\Gamma_0\cup\Gamma_L} \Big[\big(h_\alpha^s\, v_\alpha + h_3^s\, \underline{v}_3\big)\, H_2(\theta) - h_3^s\, \theta_\alpha\, \partial_3 v_\alpha\, H_1(\theta)\Big]\, d(\Gamma_0\cup\Gamma_L),\\
H_3^s(v) &= -\int_{\Gamma_0\cup\Gamma_L} h_3^s\, \theta_\alpha\, \partial_3 v_\alpha\, H_2(\theta)\, d(\Gamma_0\cup\Gamma_L),
\end{aligned}
\tag{3.10}
$$

where $H_1(\theta)$ and $H_2(\theta)$ are bounded scalar functions of θ.

3.2. The problem $\mathbf{P_s}$ formulated in $\overline{\boldsymbol{\Omega}} \times [\mathbf{0},\mathbf{T}]$. As a direct consequence of (3.6) we can formulate, for each $s \in [0,\delta]$, the problem (P_s) in the fixed domain $\overline{\Omega} \times [0,T]$, as explained in the following theorem. The new equivalent problem is denoted by (P^s), with upper index s.

THEOREM 3.1 (problem (P^s)). *For each $s \in [0,\delta]$, we assume that $d^s(x,0) = \hat{d}(x)$ in $\overline{\Omega}$, and $\hat{d}$ is independent of s. Then, the problem (P^s), for $s \neq 0$, is equivalent to the following problem (P^s) posed in the domain $\overline{\Omega} \times [0,T]$ independent of s:*

$$
\left[
\begin{array}{l}
\textit{Find } u^s : \overline{\Omega} \times [0,T] \to \mathbb{R}^3 \quad \textit{and} \quad d^s : \overline{\Omega} \times [0,T] \to \mathbb{R} \quad \textit{such that}\\
u^s(.,t) \in V(\Omega)/\mathcal{R},\\
\left[
\begin{array}{l}
a_0^s(u^s,v) + s\, a_1^s(u^s,v) + s^2\, a_2^s(u^s,v) + s^3\, a_3^s(u^s,v) + s^4\, a_4^s(u^s,v)\\
= F_0^s(v) + sF_1^s(v) + s^2\, F_2^s(v) + s^3\, F_3^s(v)\\
\quad + G_0^s(v) + s\, G_1^s(v) + s^2\, G_2^s(v) + s^3\, G_3^s(v)\\
\qquad + H_0^s(v) + s\, H_1^s(v) + s^2\, H_2^s(v) + s^3\, H_3^s(v) \qquad \forall v \in V(\Omega)/\mathcal{R},
\end{array}
\right.\\
\dot{d}^s = c(d^s) e_{33}(u^s) + a(d^s) - s\, c(d^s) \theta_\alpha \partial_{33} u_\alpha^s \quad \textit{in} \quad \Omega \times (0,T),\\
d^s(x,0) = \hat{d}(x) \quad \textit{in} \quad \overline{\Omega},
\end{array}
\right.
\tag{3.11}
$$

where the sets $V(\Omega)$ and $V(\Omega)/\mathcal{R}$ are defined by (2.10) and (2.11), for $s = 0$. We also denote by (u,d) the unique solution of problem (P^s) for $s = 0$; that is, (u,d) is *the solution of the following problem* (P^0) *(cf.* (P_s)*, for the case* $s = 0$*), formulated in* $\overline{\Omega} \times [0,T]$*:*

$$\left[\begin{array}{l} \textit{Find } u : \overline{\Omega} \times [0,T] \to \mathbb{R}^3 \quad \textit{and} \quad d : \overline{\Omega} \times [0,T] \to \mathbb{R} \quad \textit{such that} \\ u(.,t) \in V(\Omega)/\mathcal{R}, \\ a_0(u,v) = L_0(v) \equiv F_0(v) + G_0(v) + H_0(v) \qquad \forall v \in V(\Omega)/\mathcal{R}, \\ \dot{d} = c(d) e_{33}(u) + a(d) \quad \textit{in} \quad \Omega \times (0,T), \\ d(x,0) = \hat{d}(x) \quad \textit{in} \quad \overline{\Omega}, \end{array}\right. \tag{3.12}$$

where $a_0(.,.)$*,* $F_0(.)$*,* $G_0(.)$*, and* $H_0(.)$ *are independent of* s *and are defined by*

$$\begin{aligned} a_0(z,v) &= \int_\Omega \frac{1}{b_{3333}(d)}\, e_{33}(z)\, e_{33}(v)\, d\Omega \quad (a_0(.,.) \textit{ depends on } d), \\ F_0(v) &= \int_\Omega \gamma(\xi_0 + P_\eta(d))\, (f_\alpha\, v_\alpha + f_3\, \underline{v}_3)\, d\Omega \quad (F_0(.) \textit{ depends on } d), \\ G_0(v) &= \int_\Gamma \big(g_\alpha\, v_\alpha + g_3\, \underline{v}_3\big)\, d\Gamma, \\ H_0(v) &= \int_{\Gamma_0 \cup \Gamma_L} \big(h_\alpha\ v_\alpha + h_3\, \underline{v}_3\big)\, d(\Gamma_0 \cup \Gamma_L) \quad \forall z,\, v \in V(\Omega)/\mathcal{R}. \end{aligned} \tag{3.13}$$

4. Calculus and identification of the shape semiderivative. In this section we first present some preliminary estimates, which prove that the sequences $\{(u^s, d^s)\}$, $\{e_{33}(u^s)\}$, and $\{(\frac{u^s-u}{s}, \frac{d^s-d}{s})\}$ are bounded, independently of s, in appropriate functional spaces involving time. These results guarantee the existence, for each t, of a pair $(\bar{u}, \bar{d})(.,t)$, which is the weak limit of a subsequence of $\{(\frac{u^s-u}{s}, \frac{d^s-d}{s})(.,t)\}$ when $s \to 0^+$. Moreover, using the regularity of u^s, we also show that the sequences $\{e_{33}(u^s - u)\}$ and $\{d^s - d\}$ converge strongly to 0, in $C^0([0,T]; C^0(\overline{\Omega}))$, when $s \to 0^+$. These two strong convergences and the preliminary estimates are the key results that enable us to prove that the weak shape semiderivative $(\bar{u}, \bar{d})$ exists and is the unique solution of a nonlinear problem.

4.1. Preliminary estimates. We present several estimates that are needed for the identification of the shape semiderivative.

THEOREM 4.1 (first estimates for the sequences u^s and d^s). *We suppose that the conditions* (2.7)–(2.8) *are verified and* $\hat{d}(x) \in L^2(\Omega)$*; then*

$$\exists c_1 > 0 : \quad \|u^s\|_{C^0([0,T];V(\Omega)/\mathcal{R})} \le c_1 \quad \forall s \in [0,\delta], \tag{4.1}$$

$$\exists c_2 > 0 : \quad \|d^s\|_{C^0([0,T];L^2(\Omega))} \le c_2 \quad \forall s \in [0,\delta], \tag{4.2}$$

where c_1 *and* c_2 *are constants independent of* s*.*

Proof. The pair (u^s, d^s) is the solution of problem $(\mathrm{P}^{\mathrm{s}})$ (cf. (3.11)); thus for each time t,

$$a_0^s(u^s, u^s) = L^s(u^s) - \sum_{i=1}^{4} s^i\, a_i^s(u^s, u^s). \tag{4.3}$$

By the ellipticity of $a_0^s(.,.)$ in $V(\Omega)/\mathcal{R}$ (cf. (2.16)) we have for each t

$$a_0^s(u^s, u^s) \geq c\,\|u^s(.,t)\|^2_{V(\Omega)/\mathcal{R}}, \tag{4.4}$$

where c is a constant independent of s and t. From (2.15), for $s = 0$, we obtain

$$\begin{aligned} &\|\partial_3 \underline{u}_3^s(.,t)\|_{L^2(0,L)} \leq \|e_{33}(u^s(.,t))\|_{L^2(\Omega)} = \|u^s(.,t)\|_{V(\Omega)/\mathcal{R}}, \\ &\|\partial_{33} u_\alpha^s(.,t)\|_{L^2(0,L)} \leq c\,\|e_{33}(u^s(.,t))\|_{L^2(\Omega)} = c\,\|u^s(.,t)\|_{V(\Omega)/\mathcal{R}}, \end{aligned} \tag{4.5}$$

where c is a constant independent of s and t. Thus using (4.5), we easily check that $a_i^s(.,.)$, for $i = 1, 2, 3, 4$, are continuous bilinear forms that verify, for each t,

$$a_i^s(u^s, u^s) \leq c_i\,\|u^s(.,t)\|^2_{V(\Omega)/\mathcal{R}} \tag{4.6}$$

with c_i strictly positive constants, independent of s and t, and depending on θ. Using the definition of the linear form $L^s(.)$ and (4.5), we also have, for each t,

$$L^s(u^s) \leq c_L\,\|u^s(.,t)\|_{V(\Omega)/\mathcal{R}}, \tag{4.7}$$

where c_L is another strictly positive constant independent of s and t. So applying (4.4), (4.6), and (4.7), we conclude that, for each time t,

$$\left(c - \sum_{i=1}^{4} s^i\, c_i\right) \|u^s(.,t)\|^2_{V(\Omega)/\mathcal{R}} \leq c_L\,\|u^s(.,t)\|_{V(\Omega)/\mathcal{R}}, \tag{4.8}$$

and we obtain the estimate (4.1), since s is a very small parameter.

Taking now the integral in time in the remodeling rate equation of problem (3.11), we get

$$d^s(x,t) = \int_0^t \left[c(d^s)e_{33}(u^s) + a(d^s) - s\,c(d^s)\theta_\alpha \partial_{33} u_\alpha^s\right] dr + \hat{d}(x). \tag{4.9}$$

But as the material and remodeling coefficients $c(d^s)$ and $a(d^s)$ appearing in (4.9) are bounded (cf. (2.8)), we deduce that

$$\|d^s(.,t)\|_{L^2(\Omega)} \leq \int_0^T \left[c_1\,\|u^s(.,r)\|_{V(\Omega)/\mathcal{R}} + c_2\right] dr + \|\hat{d}(x)\|_{L^2(\Omega)}, \tag{4.10}$$

with c_1 and c_2 two strictly positive constants independent of s. Therefore and because of (4.1) we have the inequality (4.2). $\square$

THEOREM 4.2 (second estimate for the sequence u^s). *We assume that the hypotheses of Theorem* 2.1 *are satisfied, and* $\frac{1}{b_{3333}(d^s)} = b + \mathcal{O}(s)$, *where* $b : \mathbb{R} \to \mathbb{R}$ *is a scalar function independent of* s, $0 < |b| \leq c$ *with* $c > 0$ *a constant, and* $\mathcal{O}(s)$ *is a term of order* s *(cf. Monnier and Trabucho* [10, formulas (6) and (2)], *for a justification of this latter condition on the material coefficient* $b_{3333}(d^s)$*). Then*

$$\exists c_1 > 0 : \quad \|u^s\|_{C^0([0,T];W^{2,2}(\Omega))} \leq c_1, \tag{4.11}$$

$$\exists c_2 > 0 : \quad \|e_{33}(u^s)\|_{C^0([0,T];C^0(\overline{\Omega}))} \leq c_2, \tag{4.12}$$

where c_1 *and* c_2 *are constants independent of* $s \in [0, \delta]$.

Proof. Using (2.21)–(2.22) (in the proof of Theorem 2.1), we infer that, for each t, $\|u^s(.,t)\|_{W^{2,2}(\Omega)} \leq C(M^s; p_i^s)$, where $M^s = M_s \circ Q_s$, $p_i^s = p_{si} \circ Q_s$, and $C(M^s; p_i^s)$ is a strictly positive constant depending on the $W^{1,2}(0,L)$ norms of the elements of M^s and p_i^s. As p_i^s are data of the problem, related to the forces (cf. (2.20)), for $i = 1, 2, 3$, and due to the definition of M^s and the additional hypothesis for $\frac{1}{b_{3333}(d^s)}$, we easily deduce that there exists a constant $c > 0$, independent of s and t, such that $C(M^s; p_i^s) \leq c$ for all $s \in [0, \delta]$, and therefore we have (4.11).

Also from the regularity Theorem 2.1 we have, for each t,

$$e_{33}(u^s)(.,t) = \partial_3 \underline{u}_3^s(.,t) - x_\alpha \partial_{33} u_\alpha^s(.,t) \in W^{1,2}(\Omega) \cap C^0(\overline{\Omega}) \tag{4.13}$$

because $\partial_3 \underline{u}_3^s(.,t)$ and $\partial_{33} u_\alpha^s(.,t)$ belong to $W^{1,2}(0,L)$, which is compactly embedded in the space $C^0([0,L])$. Hence we get

$$\|e_{33}(u^s)(.,t)\|_{C^0(\overline{\Omega})} \leq c_1 \|e_{33}(u^s)(.,t)\|_{W^{1,2}(\Omega)} \leq c_2 \|u^s(.,t)\|_{W^{2,2}(\Omega)} \leq c_3, \tag{4.14}$$

where c_1, c_2, and c_3 are constants independent of s and t, and consequently we have (4.12). ∎

Theorem 4.3 (estimate for the sequence $\frac{u^s-u}{s}$). *Let (u^s, d^s) and (u, d) be the solutions of problems* (P^s) *(cf. (3.11)) and* (P^0) *(cf. (3.12)), respectively. We assume that conditions* (2.7)–(2.8) *are verified, and, for each s, $\xi_0^s = \xi_0$, $f_i^s = f_i$, $g_i^s = g_i$, $h_i^s = h_i$, where ξ_0, f_i, g_i, and h_i are independent of s. Then,*

$$\left\| \frac{u^s - u}{s} \right\|_{C^0([0,T]; V(\Omega)/\mathcal{R})} \leq c_1 \left\| \frac{d^s - d}{s} \right\|_{C^0([0,T]; L^2(\Omega))} + c_2, \tag{4.15}$$

where c_1 and c_2 are strictly positive constants independent of s and t.

Proof. In this proof we sometimes write u^s instead of $u^s(.,t)$ in order to simplify the notations. For each $t \in [0, T]$, we have

$$\frac{1}{s}[a^s(u^s, v) - a_0(u, v)] = \frac{L^s(v) - L_0(v)}{s} \qquad \forall v \in V(\Omega)/\mathcal{R}. \tag{4.16}$$

Developing this last equation for the choice $v = \frac{u^s-u}{s}$, we obtain that, for each t,

$$\left[\begin{array}{l}
a_0^s(\frac{u^s-u}{s}, \frac{u^s-u}{s}) = \frac{1}{s}\Big[F_0^s(\frac{u^s-u}{s}) - F_0(\frac{u^s-u}{s})\Big] - s^3\, a_4^s(u^s, \frac{u^s-u}{s}) \\
- \int_\Omega \frac{b_{3333}(d) - b_{3333}(d^s)}{s} \big(b_{3333}(d^s)\, b_{3333}(d)\big)^{-1} e_{33}(u) e_{33}(\frac{u^s-u}{s})\, d\Omega \\
+ \frac{1}{s}\Big[G_0^s(\frac{u^s-u}{s}) - G_0(\frac{u^s-u}{s}) + H_0^s(\frac{u^s-u}{s}) - H_0(\frac{u^s-u}{s})\Big] \\
- a_1^s(u^s, \frac{u^s-u}{s}) + F_1^s(\frac{u^s-u}{s}) + G_1^s(\frac{u^s-u}{s}) + H_1^s(\frac{u^s-u}{s}) \\
+ s\,\Big[- a_2^s(u^s, \frac{u^s-u}{s}) + F_2^s(\frac{u^s-u}{s}) + G_2^s(\frac{u^s-u}{s}) + H_2^s(\frac{u^s-u}{s})\Big] \\
+ s^2\,\Big[- a_3^s(u^s, \frac{u^s-u}{s}) + F_3^s(\frac{u^s-u}{s}) + G_3^s(\frac{u^s-u}{s}) + H_3^s(\frac{u^s-u}{s})\Big].
\end{array}\right. \tag{4.17}$$

Using this last equation, the ellipticity of $a_0^s(.,.)$, and the properties of continuity of all the other remaining terms in (4.17), we obtain the estimate (4.15). We next explain these calculations in detail, analyzing (4.17) for each t.

Because of condition (2.7), we have for each t,

$$\left| a_0^s\left(\frac{u^s-u}{s}, \frac{u^s-u}{s}\right)\right| \geq c \left\|\frac{u^s-u}{s}(.,t)\right\|^2_{V(\Omega)/\mathcal{R}}, \tag{4.18}$$

where c is a strictly positive constant independent of s and t. Using the definitions of $a_i^s(.,.)$ and the estimate (4.1), we obviously obtain

$$\left|a_i^s\left(u^s, \frac{u^s-u}{s}\right)\right| \leq c_{a_i} \left\|\frac{u^s-u}{s}(.,t)\right\|_{V(\Omega)/\mathcal{R}}, \tag{4.19}$$

where c_{a_i}, for $i = 1, 2, 3, 4$, are strictly positive constants independent of s and t. Considering now the definitions of the forms F_i^s, G_i^s, and H_i^s, associated with the applied forces, we easily check that, for each t,

$$F_i^s\left(\frac{u^s-u}{s}\right) + G_i^s\left(\frac{u^s-u}{s}\right) + H_i^s\left(\frac{u^s-u}{s}\right) \leq c_i \left\|\frac{u^s-u}{s}(.,t)\right\|_{V(\Omega)/\mathcal{R}}, \tag{4.20}$$

where c_i are strictly positive constants independent of s and t, for $i = 1, 2, 3$. In addition we also have, for each t, $G_0^s(\frac{u^s-u}{s}) = G_0(\frac{u^s-u}{s})$ and $H_0^s(\frac{u^s-u}{s}) = H_0(\frac{u^s-u}{s})$. Using the mean value theorem for the operator P_η, we deduce

$$\left[\begin{array}{l} \left|\frac{1}{s}\left[F_0^s(\frac{u^s-u}{s}) - F_0(\frac{u^s-u}{s})\right]\right| \\ \quad \leq \int_\Omega \gamma \left|\frac{P_\eta(d^s)-P_\eta(d)}{d^s-d}\right| \left|\frac{d^s-d}{s}\right| \left|f_\alpha \, \frac{u_\alpha^s-u_\alpha}{s} + f_3 \, \frac{u_3^s-u_3}{s}\right| d\Omega \\ \quad\quad \leq c_0 \, \|\frac{d^s-d}{s}(.,t)\|_{L^2(\Omega))} \, \|\frac{u^s-u}{s}(.,t)\|_{V(\Omega)/\mathcal{R}}, \end{array}\right. \tag{4.21}$$

where c_0 is a strictly positive constant independent of s and t. Finally, using the mean value theorem for the material coefficient $b_{3333}(.)$, the estimate (4.12) for $s = 0$, and condition (2.7), we get

$$\left[\begin{array}{l} \left|\int_\Omega \frac{b_{3333}(d)-b_{3333}(d^s)}{s} \big(b_{3333}(d^s)\, b_{3333}(d)\big)^{-1} e_{33}(u)e_{33}(\frac{u^s-u}{s})\, d\Omega\right| \\ = \left|\int_\Omega \frac{b_{3333}(d)-b_{3333}(d^s)}{d^s-d} \, \frac{d^s-d}{s} \big(b_{3333}(d^s)\, b_{3333}(d)\big)^{-1} e_{33}(u)e_{33}(\frac{u^s-u}{s})\, d\Omega\right| \\ \quad\quad \leq c_b \, \|\frac{d^s-d}{s}(.,t)\|_{L^2(\Omega)} \, \|\frac{u^s-u}{s}(.,t)\|_{V(\Omega)/\mathcal{R}}, \end{array}\right. \tag{4.22}$$

where c_b is a strictly positive constant independent of s and t. Therefore using (4.17) and the estimates (4.18)–(4.22), we have, for each t,

$$\left\{\begin{array}{l} c\, \|\frac{u^s-u}{s}(.,t)\|^2_{V(\Omega)/\mathcal{R}} \\ \leq c_1 \, \|\frac{d^s-d}{s}(.,t)\|_{L^2(\Omega)} \, \|\frac{u^s-u}{s}(.,t)\|_{V(\Omega)/\mathcal{R}} + c_2 \, \|\frac{u^s-u}{s}(.,t)\|_{V(\Omega)/\mathcal{R}}, \end{array}\right. \tag{4.23}$$

where c, c_1, and c_2 are strictly positive constants independent of s and t. The proof is finished, dividing (4.23) by $c\, \|\frac{u^s-u}{s}(.,t)\|_{V(\Omega)/\mathcal{R}}$. ∎

THEOREM 4.4 (estimate for the sequence $\frac{d^s-d}{s}$). *Let* (u^s, d^s) *and* (u, d) *be the solutions of problems* (3.11) *and* (3.12), *respectively. We assume that the hypotheses of Theorem* 4.3 *are satisfied. Then*

$$\left\|\frac{d^s-d}{s}\right\|_{C^0([0,T];L^2(\Omega))} \leq c, \tag{4.24}$$

where c is a strictly positive constant independent of s and t.

Proof. Subtracting the remodeling rate equations of problems (P^s) and (P^0) (cf. (3.11) and (3.12)), taking the integral in time between 0 and t and then the

$L^2(\Omega)$ norm, using the conditions (2.7)–(2.8) and the mean value theorem for the terms $c(d^s) - c(d)$ and $a(d^s) - a(d)$, we obtain, for each t, the estimate

$$\left\{\begin{array}{l} \|(d^s - d)(.,t)\|_{L^2(\Omega)} \\ \leq \int_0^t \Big[c_1 \,\|e_{33}(u^s - u)(.,r)\|_{L^2(\Omega)} + s\, c_4 \,\|\partial_{33} u^s_\alpha(.,r)\|_{L^2(\Omega)} \\ \qquad + \big(c_2 \,\|e_{33}(u)(.,r)\|_{C^0(\overline{\Omega})} + c_3\big)\, \|(d^s - d)(.,r)\|_{L^2(\Omega)}\Big]\, dr, \end{array}\right. \tag{4.25}$$

where c_1, c_2, c_3, and c_4 are strictly positive constants independent of s and t. However,

$$\begin{aligned} &\|e_{33}(u^s - u)(.,t)\|_{L^2(\Omega)} = \|(u^s - u)(.,t)\|_{V(\Omega)/\mathcal{R}}, \\ &\|e_{33}(u)(.,t)\|_{C^0(\overline{\Omega})} \leq c_2, \\ &\|\partial_{33} u^s_\alpha(.,t)\|_{L^2(\Omega)} \leq c_0 \|e_{33} u^s(.,t)\|_{L^2(\Omega)} = c_0 \|u^s(.,t)\|_{V(\Omega)/\mathcal{R}} \leq c, \end{aligned} \tag{4.26}$$

where c_2 is the constant defined in (4.12) for the case $s = 0$, c_0 is defined in (4.5), and c is a constant depending on the constant defined in (4.1); these three constants are independent of s and t. So, dividing (4.25) by s, we have from (4.25)–(4.26) and Theorem 4.3

$$\left\|\frac{d^s - d}{s}(.,t)\right\|_{L^2(\Omega)} \leq c_5 + \int_0^t c_6 \left\|\frac{d^s - d}{s}(.,r)\right\|_{L^2(\Omega)} dr, \tag{4.27}$$

where c_5 and c_6 are strictly positive constants independent of s and t. Then, applying the integral Gronwall inequality (cf. Evans [6, p. 625]),

$$\left\|\frac{d^s - d}{s}(.,t)\right\|_{L^2(\Omega)} \leq c_5\,(1 + t\, c_6\, e^{c_6\, t}) \qquad \forall t \in [0,T], \tag{4.28}$$

which implies (4.24). ◻

Corollary 4.5. *With the hypotheses of Theorem* 4.3*,*

$$\exists c > 0 : \qquad \left\|\frac{u^s - u}{s}\right\|_{C^0([0,T];V(\Omega)/\mathcal{R})} \leq c, \tag{4.29}$$

where c is independent of s and t.

Thus we conclude that the solutions (u^s, d^s) and (u, d), of problems (P^s) and (P^0), verify for all $s \in [0, \delta]$

$$\left\|\frac{u^s - u}{s}\right\|_{C^0([0,T];V(\Omega)/\mathcal{R})} \leq c_1 \quad \text{and} \quad \left\|\frac{d^s - d}{s}\right\|_{C^0([0,T];L^2(\Omega))} \leq c_2, \tag{4.30}$$

where c_1 and c_2 are strictly positive constants independent of s. As a consequence of this property we state the following theorem.

Theorem 4.6 (weak limits of the quotient sequences). *Let (u^s, d^s) and (u, d) be the solutions of problems (P^s) and (P^0), and assume that the hypotheses of Theorem 4.3 are verified. Then, for each t, there exists a subsequence of $\{(u^s, d^s)(.,t)\}$, also denoted by $\{(u^s, d^s)(.,t)\}$, and elements $\bar{u}(.,t) \in V(\Omega)/\mathcal{R}$ and $\bar{d}(.,t) \in L^2(\Omega)$ such* *that, when the parameter* $s \to 0^+$,

$$\frac{u^s - u}{s}(.,t) \rightharpoonup \bar{u}(.,t) \quad \textit{weakly in} \quad V(\Omega)/\mathcal{R}, \tag{4.31}$$

$$e_{33}\left(\frac{u^s - u}{s}\right)(.,t) \rightharpoonup e_{33}(\bar{u})(.,t) \quad \textit{weakly in} \quad L^2(\Omega), \tag{4.32}$$

$$\frac{d^s - d}{s}(.,t) \rightharpoonup \bar{d}(.,t) \quad \textit{weakly in} \quad L^2(\Omega). \tag{4.33}$$

Therefore, when $s \to 0^+$, $(u^s - u)(.,t)$ *converges strongly to* 0 *in* $V(\Omega)/\mathcal{R}$, *and the sequences* $e_{33}(u^s - u)(.,t)$ *and* $(d^s - d)(.,t)$ *converge strongly to* 0 *in* $L^2(\Omega)$.

We conclude this section with a convergence result concerning the sequences $\{u^s\}$ and $e_{33}(u^s)$, and a corollary about the convergence of $\{d^s\}$, which will be useful in subsection 4.2.

THEOREM 4.7 (strong limit of $e_{33}(u^s)$). *Let* (u^s, d^s) *and* (u, d) *be the solutions of problems* (P^s) *and* (P^0) *and assume that the hypotheses of Theorems* 2.1, 4.2, *and* 4.3 *are verified. Then there exists a subsequence of* $\{u^s\}$, *also denoted by* $\{u^s\}$, *that verifies the following convergence, when the parameter* $s \to 0^+$:

$$e_{33}(u^s - u) \longrightarrow 0 \quad \textit{strongly in} \quad C^0([0,T]; C^0(\overline{\Omega})). \tag{4.34}$$

Proof. Recalling the definition of $u^s - u$ and its regularity (cf. Theorem 2.1), we have, for each t, $(u^s_\alpha - u_\alpha)(.,t) \in W^{3,2}(0,L)$ for $\alpha = 1,2$ and $(\underline{u}^s_3 - \underline{u}_3)(.,t) \in W^{2,2}(0,L)$. The calculus of $e_{33}(u^s - u)$ gives

$$e_{33}(u^s - u) = \partial_3(u^s_3 - u_3) = \partial_3(\underline{u}^s_3 - \underline{u}_3) - x_\alpha \partial_{33}(u^s_\alpha - u_\alpha), \tag{4.35}$$

where $[\partial_3(\underline{u}^s_3 - \underline{u}_3)](.,t) \in W^{1,2}(0,L)$ and $[\partial_{33}(u^s_\alpha - u_\alpha)](.,t) \in W^{1,2}(0,L)$, for $\alpha = 1,2$. But, because of the strong convergence of $e_{33}(u^s - u)(.,t)$ to 0 in $L^2(\Omega)$ (cf. Theorem 4.6), and applying the definition of $\|e_{33}(u^s - u)(.,t)\|_{L^2(\Omega)}$ (cf. (2.15)), we conclude immediately that, for each t, $\partial_3(\underline{u}^s_3 - \underline{u}_3)(.,t)$ and $\partial_{33}(u^s_\alpha - u_\alpha)(.,t)$ converge strongly to 0 in $L^2(0,L)$. On the other hand, we get directly from (4.11)

$$\left\{ \begin{aligned} &\|e_{33}(u^s - u)\|_{C^0([0,T];W^{1,2}(\Omega))} \le \|u^s - u\|_{C^0([0,T];W^{2,2}(\Omega))} \\ &\qquad \le \|u^s\|_{C^0([0,T];W^{2,2}(\Omega))} + \|u\|_{C^0([0,T];W^{2,2}(\Omega))} \le c, \end{aligned} \right. \tag{4.36}$$

where c is a constant independent of s. Therefore the sequences $\partial_3(\underline{u}^s_3 - \underline{u}_3)$ and $\partial_{33}(u^s_\alpha - u_\alpha)$ are bounded in $C^0([0,T]; W^{1,2}(0,L))$, and consequently, we obtain that $\partial_3(\underline{u}^s_3 - \underline{u}_3)(.,t)$ and $\partial_{33}(u^s_\alpha - u_\alpha)(.,t)$ weakly converge to 0, in $W^{1,2}(0,L)$ when $s \to 0^+$. But as the space $W^{1,2}(0,L)$ is compactly embedded in $C^0([0,L])$, we have that $\partial_3(\underline{u}^s_3 - \underline{u}_3)(.,t)$ and $\partial_{33}(u^s_\alpha - u_\alpha)(.,t)$ strongly converge to 0 in $C^0([0,L])$ when $s \to 0^+$. This implies the strong convergence of $e_{33}(u^s - u)$ to 0 in $C^0([0,T]; C^0(\overline{\Omega}))$. ∎

COROLLARY 4.8 (strong limit of d^s). *Let* (u^s, d^s) *and* (u, d) *be the solutions of problems* (P^s) *and* (P^0), *and assume the same hypotheses of Theorem* 4.7. *Then there exists a constant* $c > 0$ *independent of* s *such that when* $s \to 0^+$

$$d^s - d \longrightarrow 0 \quad \textit{strongly in} \quad C^0([0,T]; C^0(\overline{\Omega})). \tag{4.37}$$

Proof. Using exactly the same arguments as in the beginning of the proof of Theorem 4.4,

$$|(d^s - d)(x,t)| \leq \int_0^t \Big[c_1|e_{33}(u^s - u)(x,r)| + c_2|(d^s - d)(x,r)| + sc_3\Big]dr, \tag{4.38}$$

where the constants c_i, for $i = 1,2,3$, are strictly positive constants independent of s and t, and consequently

$$\left\{\begin{aligned} &|(d^s - d)(x,t)| \leq \textstyle\int_0^t c_2\,|(d^s - d)(x,r)|\,dr \\ &\qquad + \underbrace{T\Big[c_1\,\|e_{33}(u^s - u)\|_{C^0([0,T];C^0(\overline{\Omega}))} + s\,c_3\Big]}_{\varphi^s}. \end{aligned}\right. \tag{4.39}$$

Because of the strong convergence (4.34), the scalar $\varphi^s \to 0$ when $s \to 0^+$. Then we obtain the convergence (4.37), applying to (4.39) the integral Gronwall inequality (cf. Evans [6, p. 625]). ∎

4.2. Shape semiderivatives. The objective of this section is to identify, for each t, the weak limits $\bar{u}(.,t)$ and $\bar{d}(.,t)$ of the sequences $\{\frac{u^s-u}{s}(.,t)\}$ and $\{\frac{d^s-d}{s}(.,t)\}$, defined in (4.31) and (4.33). The procedure is the following: we subtract and divide by s the equilibrium variational equations and the remodeling rate equations in problems (P^s) and (P^0), and then we take the limit, when the parameter $s \to 0^+$. We conclude that, for each t, the pair $(\bar{u}(.,t), \bar{d}(.,t))$ is the solution of another nonlinear problem. Finally we end up proving that $(\bar{u}, \bar{d})$ is the unique solution of this latter problem in the space $C^1([0,T]; V(\Omega)/\mathcal{R}) \times C^1([0,T]; C^0(\overline{\Omega}))$.

4.2.1. Weak limit $\bar{\mathbf{u}}(.,\mathbf{t})$. Subtracting and dividing by s the two equilibrium variational equations of problems (P^s) and (P^0), we obtain, for each $t \in [0,T]$ (cf. (4.16)–(4.17)),

$$\left[\begin{aligned} &\textstyle\int_\Omega \frac{1}{b_{3333}(d^s)} e_{33}(\frac{u^s-u}{s}) e_{33}(v) d\Omega \\ &+ \textstyle\int_\Omega \frac{b_{3333}(d) - b_{3333}(d^s)}{s} \big(b_{3333}(d^s)\, b_{3333}(d)\big)^{-1} e_{33}(u) e_{33}(v)\, d\Omega \\ &+ a_1^s(u^s, v) - F_1^s(v) - G_1^s(v) - H_1^s(v) \\ &= \tfrac{1}{s}\big[F_0^s(v) - F_0(v) + G_0^s(v) - G_0(v) + H_0^s(v) - H_0(v)\big] \\ &+ s\,\big[-a_2^s(u^s,v) + F_2^s(v) + G_2^s(v) + H_2^s(v)\big] \\ &+ s^2\,\big[-a_3^s(v) + F_3^s(v) + G_3^s(v) + H_3^s(v)\big] - s^3\, a_4^s(u^s, v). \end{aligned}\right. \tag{4.40}$$

Computing now for each t the limit of each term of (4.40), we obtain the following theorem.

THEOREM 4.9 (identification of $\bar{u}(.,t)$). *We assume the hypotheses of Theorems 2.1, 4.2, and 4.3. Then the weak limit $\bar{u}(.,t)$ of the sequence $\{\frac{u^s-u}{s}\}(.,t)$ verifies the following variational equation for each $t \in [0,T]$:*

$$B(\bar{u}, v) = S(v) \qquad \forall v \in V(\Omega)/\mathcal{R}. \tag{4.41}$$

The linear form $S(.)$ is defined in $V(\Omega)/\mathcal{R}$ by

$$
(4.42)\quad \begin{cases} S(v) = \int_\Omega b'_{3333}(d)\, b_{3333}(d)^{-2}\, \bar{d}\, e_{33}(u)\, e_{33}(v)\, d\Omega \\ - \int_\Omega \frac{1}{b_{3333}(d)} \big[-\theta_\alpha \big(e_{33}(u)\partial_{33} v_\alpha + e_{33}(v)\partial_{33} u_\alpha \big) \\ \qquad\qquad + e_{33}(u) e_{33}(v) \operatorname{div} \theta \big] d\Omega \\ + \int_\Omega \gamma\, \bar{d}\, P'_\eta(d)\, (f_\alpha\, v_\alpha + f_3\, \underline{v}_3)\, d\Omega \\ + \int_\Omega \gamma (\xi_0 + P_\eta(d)) \big[(f_\alpha\, v_\alpha + f_3\, \underline{v}_3) \operatorname{div}\theta - f_3\, \theta_\alpha\, \partial_3 v_\alpha \big] d\Omega \\ + \int_\Gamma \big[(g_\alpha\, v_\alpha + g_3\, \underline{v}_3)\, G_1(\theta, n) - g_3\, \theta_\alpha\, \partial_3 v_\alpha \big] d\Gamma \\ + \int_{\Gamma_0 \cup \Gamma_L} \big[(h_\alpha\, v_\alpha + h_3\, \underline{v}_3)\, H_1(\theta) - h_3\, \theta_\alpha\, \partial_3 v_\alpha \big] d\Gamma_0 \cup \Gamma_L, \end{cases}
$$

with $b'_{3333}(.)$ and $P'_\eta(.)$ the first derivatives of the scalar functions $b_{3333}(.)$ and $P_\eta(.)$, respectively. The bilinear form $B(.,.)$ is defined by

$$
(4.43)\qquad B(z, v) = \int_\Omega \frac{1}{b_{3333}(d)}\, e_{33}(z)\, e_{33}(v)\, d\Omega \qquad \forall z,\ v \in V(\Omega)/\mathcal{R}.
$$

Proof. We give a sketch of the computation of the limits in (4.40), for each t.

The first term in (4.40) verifies, when $s \to 0^+$,

$$
(4.44)\qquad \int_\Omega \frac{1}{b_{3333}(d^s)} e_{33}\left(\frac{u^s - u}{s}\right) e_{33}(v) d\Omega \ \longrightarrow\ \int_\Omega \frac{1}{b_{3333}(d)} e_{33}(\bar{u}) e_{33}(v) d\Omega,
$$

because the sequence $e_{33}(\frac{u^s-u}{s})(.,t)$ weakly converges to $e_{33}(\bar{u})(.,t)$ in $L^2(\Omega)$ (cf. (4.32)), and $\frac{e_{33}(v)}{b_{3333}(d^s)}$ converges strongly to $\frac{e_{33}(v)}{b_{3333}(d)}$ in $L^2(\Omega)$, due to the condition (2.7), the mean value theorem for the function $\frac{1}{b_{3333}(.)}$, and the strong convergence of d^s to d in $C^0([0,T]; C^0(\overline{\Omega}))$ (cf. (4.37)).

The second term in (4.40) converges to

$$
(4.45)\qquad -\int_\Omega b'_{3333}(d)\, b_{3333}(d)^{-2}\, \bar{d}\, e_{33}(u)\, e_{33}(v)\, d\Omega,
$$

when $s \to 0^+$, because of condition (2.7), the mean value theorem for the function $b_{3333}(.)$, the condition (4.12) for the case $s = 0$, the strong convergence of d_s to d in $C^0([0,T]; C^0(\overline{\Omega}))$ (cf. Corollary 4.8, formula (4.37)), and the weak convergence of $\frac{d^s-d}{s}(.,t)$ to $\bar{d}(.,t)$ in $L^2(\Omega)$.

For the third term in (4.40), we have that $a_1^s(u^s, v)$ converges to

$$
(4.46)\qquad \int_\Omega \frac{1}{b_{3333}(d)} \Big[-\theta_\alpha \big(e_{33}(u)\partial_{33} v_\alpha + e_{33}(v)\partial_{33} u_\alpha \big) + e_{33}(u) e_{33}(v) \operatorname{div}\theta \Big] d\Omega
$$

when $s \to 0^+$, and $F_1^s(v) + G_1^s(v) + H_1^s(v)$ converges to

$$
(4.47)\quad \begin{cases} \int_\Omega \gamma(\xi_0 + P_\eta(d)) \Big[(f_\alpha\, v_\alpha + f_3\, \underline{v}_3) \operatorname{div}\theta - f_3\, \theta_\alpha\, \partial_3 v_\alpha \Big] d\Omega \\ + \int_\Gamma \Big[(g_\alpha\, v_\alpha + g_3\, \underline{v}_3)\, G_1(\theta, n) - g_3\, \theta_\alpha\, \partial_3 v_\alpha \Big] d\Gamma \\ + \int_{\Gamma_0 \cup \Gamma_L} \Big[(h_\alpha\, v_\alpha + h_3\, \underline{v}_3)\, H_1(\theta) - h_3\, \theta_\alpha\, \partial_3 v_\alpha \Big] d\Gamma_0 \cup \Gamma_L. \end{cases}
$$

To prove (4.46) we remark that, when $s \to 0^+$, $\|\big(e_{33}(u^s) - e_{33}(u)\big)(.,t)\|_{L^2(\Omega)}$ and $\|(d^s - d)(.,t)\|_{L^2(\Omega)}$ converge to 0 (cf. Theorem 4.6). To obtain (4.47) we apply the definitions of the forms F_1^s, G_1^s, H_1^s and the strong convergence of $P_\eta(d^s)$ to $P_\eta(d)$, when $s \to 0^+$, in the space $C^0([0,T]; C^0(\overline{\Omega}))$.

The fourth term in (4.40) converges to

$$\int_\Omega \gamma\, \bar{d}\, P'_\eta(d)\, (f_\alpha\, v_\alpha + f_3\, \underline{v}_3)\, d\Omega \tag{4.48}$$

when $s \to 0^+$. In fact, we have $G_0^s(v) = G_0(v)$ and $H_0^s(v) = H_0(v)$, and

$$\begin{cases} \frac{1}{s}\Big[F_0^s(v) - F_0(v)\Big] - \int_\Omega \gamma\, \bar{d}\, P'_\eta(d)\, (f_\alpha\, v_\alpha + f_3\, \underline{v}_3)\, d\Omega \\ = \int_\Omega \gamma \Big[\frac{P_\eta(d^s) - P_\eta(d)}{d^s - d} \, \big(\frac{d^s - d}{s} - \bar{d}\big) \\ \qquad + \big(\frac{P_\eta(d^s) - P_\eta(d)}{d^s - d} - P'_\eta(d)\big)\, \bar{d}\Big] (f_\alpha\, v_\alpha + f_3\, \underline{v}_3)\, d\Omega. \end{cases} \tag{4.49}$$

When $s \to 0^+$, $\gamma\, \frac{P_\eta(d^s) - P_\eta(d)}{d^s - d}$ converges strongly to $\gamma\, P'_\eta(d)$ in $C^0([0,T]; C^0(\overline{\Omega}))$, and, for each t, $\frac{d^s - d}{s}$ converges weakly to $\bar{d}$ in $L^2(\Omega)$ (cf. Theorem 4.6, formula (4.33)), and $\bar{d}\, (f_\alpha\, v_\alpha + f_3\, \underline{u}_3)$ belongs to $L^1(\Omega)$. Thus (4.49) converges to 0, for each t, when $s \to 0^+$.

Finally the last two terms in (4.40) converge to 0 when $s \to 0^+$, because those are composed of bounded terms multiplied by a positive power of s. ∎

4.2.2. Weak limit $\bar{\mathbf{d}}(.,\mathbf{t})$. By subtracting and dividing by s the remodeling rate equations of problems (P^s) and (P^0), and integrating in time between 0 and t, we obtain the following theorem.

THEOREM 4.10 (identification of $\bar{d}(.,t)$). *We assume that the hypotheses of Theorems* 2.1, 4.2, *and* 4.3 *are verified. For each t, the weak limit $\bar{d}(.,t)$ of the sequence $\{\frac{d^s - d}{s}(.,t)\}$ is the solution of the following ordinary differential equation with respect to time:*

$$\begin{cases} \dot{\bar{d}} = c(d) e_{33}(\bar{u}) + \bar{d}\, [c'(d)\, e_{33}(u) + a'(d)] - c(d)\, \theta_\alpha \partial_{33} u_\alpha, \\ \bar{d}(x, 0) = 0. \end{cases} \tag{4.50}$$

Proof. For any $v \in L^2(\Omega)$ and for each t, we have

$$\begin{cases} \int_\Omega \frac{d^s - d}{s}\, v\, d\Omega = \int_0^t \Big(\int_\Omega \Big[c(d^s)\, e_{33}(\frac{u^s - u}{s}) + \frac{c(d^s) - c(d)}{s}\, e_{33}(u) \\ \qquad\qquad + \frac{a(d^s) - a(d)}{s} - c(d^s) \theta_\alpha \partial_{33} u^s_\alpha \Big]\, v\, d\Omega \Big)\, dr. \end{cases} \tag{4.51}$$

On the other hand, for each t, and when $s \to 0^+$, $c(d^s)$ converges strongly to $c(d)$ in $C^0(\overline{\Omega})$, $e_{33}(\frac{u^s - u}{s})$ converges weakly to $e_{33}(\bar{u})$ in $L^2(\Omega)$, $\frac{c(d^s) - c(d)}{s}\, e_{33}(u)$ converges weakly to $\bar{d}\, c'(d)\, e_{33}(u)$ in $L^2(\Omega)$, $\frac{a(d^s) - a(d)}{s}$ converges weakly to $\bar{d}\, a'(d)$ in $L^2(\Omega)$, and $\partial_{33} u^s_\alpha$ converges strongly to $\partial_{33} u_\alpha$ in $L^2(\Omega)$. Hence, we have that, for each t and when $s \to 0^+$, the sequence $\int_\Omega \frac{d^s - d}{s}\, v\, d\Omega$ converges to

$$\int_\Omega \left(\int_0^t \Big[c(d) e_{33}(\bar{u}) + \bar{d}\, c'(d)\, e_{33}(u) + \bar{d}\, a'(d) - c(d)\, \theta_\alpha \partial_{33} u_\alpha \Big]\, dr \right) v\, d\Omega. \tag{4.52}$$

But by (4.33), $\frac{d^s - d}{s}(.,t)$ converges weakly to $\bar{d}(.,t)$ in $L^2(\Omega)$ when $s \to 0^+$. Therefore $\bar{d}(.,t)$ must verify (4.50), since the weak limit is unique. ∎

4.2.3. Final identification result. Assembling the results of Theorems 4.9 and 4.10, we have the next theorem, which completely identifies, for each t, the (weak) shape semiderivatives $\bar{u}(.,t)$ and $\bar{d}(.,t)$.

THEOREM 4.11. *We assume that the hypotheses of Theorems* 2.1, 4.2, *and* 4.3 *are verified. For each* $t \in [0,T]$*, the weak limit* $(\bar{u},\bar{d})(.,t)$ *is an element of the space* $(V(\Omega)/\mathcal{R}) \times L^2(\Omega)$ *and is the solution of the following nonlinear problem* $(\bar{\mathrm{P}})$:

$$\left[\begin{array}{l} \text{Find } \bar{u} : \overline{\Omega} \times [0,T] \to \mathbb{R}^3 \quad \text{and} \quad \bar{d} : \overline{\Omega} \times [0,T] \to \mathbb{R} \quad \text{such that} \\ \bar{u}(.,t) \in V(\Omega)/\mathcal{R}, \\ B(\bar{u},v) = S(v) \qquad \forall v \in V(\Omega)/\mathcal{R}, \\ \dot{\bar{d}} = c(d)e_{33}(\bar{u}) + \bar{d}\,[c'(d)\,e_{33}(u) + a'(d)] - c(d)\,\theta_\alpha \partial_{33} u_\alpha \quad \text{in } \Omega \times (0,T), \\ \bar{d}(x,0) = 0 \quad \text{in} \quad \overline{\Omega}, \end{array}\right. \tag{4.53}$$

where the linear form $S(.)$ *and the bilinear form* $B(.,.)$ *are defined by* (4.42) *and* (4.43)*, respectively. We observe that* $S(.)$ *depends on* (u,d)*, which is the solution of problem* (P^0)*, and also on* $\bar{d}$*. The bilinear form* $B(.,.)$ *depends on* d*, that is, the measure of change in volume fraction of the elastic material of problem* (P^0)*. Moreover, there exists a unique solution* $(\bar{u},\bar{d})$ *of problem* $(\bar{\mathrm{P}})$ *such that* $\bar{u} \in C^1([0,T];V(\Omega)/\mathcal{R})$ *and* $\bar{d} \in C^1([0,T];C^0(\overline{\Omega}))$*. Consequently, for each* t*, the entire sequence* $\{(\frac{u^s-u}{s}, \frac{d^s-d}{s})(.,t)\}$ *weakly converges to* $(\bar{u},\bar{d})(.,t)$ *in the space* $V(\Omega)/\mathcal{R} \times L^2(\Omega)$ *when* $s \to 0^+$*. Thus, there exists the weak shape semiderivative of the shape map* $J(\Omega_s) = (u^s,d^s)$ *at* $s = 0$ *in the direction of the vector field* θ *(cf.* (2.24)*), and it is perfectly defined, for each* t*, by* $dJ(\Omega;\theta)(.,t) = (\bar{u},\bar{d})(.,t)$*, where* $(\bar{u},\bar{d}) \in C^1([0,T];V(\Omega)/\mathcal{R}) \times C^1([0,T];C^0(\overline{\Omega}))$ *is the unique solution of problem* $(\bar{\mathrm{P}})$*.*

Proof. The arguments used to prove the existence and uniqueness of the solution $(\bar{u},\bar{d})$ to problem $(\bar{\mathrm{P}})$, in the space $C^1([0,T];V(\Omega)/\mathcal{R}) \times C^1([0,T];C^0(\overline{\Omega}))$, are analogous to those utilized in the proof of existence and uniqueness of the solution to problem $(\mathrm{P_s})$ (cf. Figueiredo and Trabucho [7]) and rely on the Schauder fixed point theorem. ∎

5. Conclusions and future work. In this paper we have considered the family $\overline{\Omega}_s$ of perturbed thin rods, for $s \in [0,\delta]$, and the corresponding family of solutions (u^s,d^s) of the nonlinear asymptotic adaptive elastic model, derived in Figueiredo and Trabucho [7]. We have proved that, for each t, the sequence $(\frac{u^s-u}{s}, \frac{d^s-d}{s})(.,t)$ converges weakly to $(\bar{u},\bar{d})(.,t)$ in the space $(V(\Omega)/\mathcal{R}) \times L^2(\Omega)$ when $s \to 0^+$. Consequently, for each t, $(\bar{u},\bar{d})(.,t)$ is the weak shape semiderivative of the function $J(\Omega_s) = (u^s,d^s)$ at $s = 0$ in the direction of the vector field θ. Moreover, we have showed that the pair $(\bar{u},\bar{d})$ is the unique solution of another nonlinear problem that couples a variational equation, depending on (u,d) and $\bar{d}$, and an ordinary differential equation with respect to time, depending on (u,d) and $\bar{u}$. We intend to apply this methodology to analyze the weak shape semiderivative of the solution to the nonlinear adaptive elastic asymptotic model (2.1)–(2.4), but for the case where the remodeling rate equation (2.4) depends nonlinearly on $e_{33}(u_s)$ (cf. Figueiredo and Trabucho [7]). We think that this nonlinear term may generate some difficulties in proving that the sequence $\{\frac{d^s-d}{s}\}$ is bounded, independently of s, and subsequently in the identification of the shape semiderivative.

REFERENCES

[1] P. G. Ciarlet, *Mathematical Elasticity* 1: *Three-Dimensional Elasticity*, Stud. Math. Appl. 20, North–Holland, Amsterdam, 1988.
[2] P. G. Ciarlet, *Introduction to Linear Shell Theory*, Gauthier-Villars, Paris, 1998.
[3] S. C. Cowin and D. H. Hegedus, *Bone remodeling* I: *Theory of adaptive elasticity*, J. Elasticity, 6 (1976), pp. 313–326.
[4] S. C. Cowin and R. R. Nachlinger, *Bone remodeling* III: *Uniqueness and stability in adaptive elasticity theory*, J. Elasticity, 8 (1978), pp. 285–295.
[5] M. C. Delfour and J. P. Zolésio, *Shapes and Geometries, Analysis, Differential Calculus, and Optimization*, Adv. Des. Control 4, SIAM, Philadelphia, 2001.
[6] L. C. Evans, *Partial Differential Equations*, AMS, Providence, RI, 1998.
[7] I. N. Figueiredo and L. Trabucho, *Asymptotic model of a nonlinear adaptive elastic rod*, Math. Mech. Solids, 9 (2004), pp. 331–354.
[8] J. Haslinger and R. A. E. Mäkinen, *Introduction to Shape Optimization, Theory, Approximation, and Computation*, Adv. Des. Control 7, SIAM, Philadelphia, 2003.
[9] D. H. Hegedus and S. C. Cowin, *Bone remodeling* II: *Small strain adaptive elasticity*, J. Elasticity, 6 (1976), pp. 337–352.
[10] J. Monnier and L. Trabucho, *An existence and uniqueness result in bone remodeling theory*, Comput. Methods Appl. Mech. Engrg., 151 (1998), pp. 539–544.
[11] L. Trabucho and J. M. Viaño, *Mathematical modelling of rods*, in Handb. Numer. Anal. 4, P. G. Ciarlet and J. L. Lions, eds., North–Holland, Amsterdam, 1996, pp. 487–974.
[12] T. Valent, *Boundary Value Problems of Finite Elasticity*, Springer Tracts Nat. Philos. 31, Springer-Verlag, New York, 1988.