



Levels of incomplete information in group decision models – A comprehensive simulation study



Rudolf Vetschera^{a,*}, Paula Sarabando^b, Luis Dias^c

^a University of Vienna, Austria

^b INESC Coimbra and Viseu Politechnical Institute, Portugal

^c INESC Coimbra and FEUC, University of Coimbra, Portugal

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ABSTRACT

We present a comprehensive computational study on the effects of providing different forms of incomplete preference information in additive group decision models. We consider different types of information on individual preferences, and on weights of the group members, and study their effects on conclusiveness, efficiency and fairness of outcomes at the group level. Furthermore, we analyze possible violations of the axiom of independence of irrelevant alternatives (IIA) as well as the impact of problem characteristics, in particular initial agreement between group members. Our results indicate that providing information in the form of a ranking of differences between consecutive alternatives comes close to providing exact cardinal preference information in several outcome dimensions. However, group decision procedures based on incomplete preference information also show a significant amount of violations of the IIA axiom.

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1. Introduction

The problem of aggregating individual preferences to a group preference on a set of alternatives has for centuries been of central importance to many fields of research such as political science, economics or decision theory. A seminal contribution to the field was made by Arrow [1], who in his famous impossibility theorem showed that it is not possible to aggregate ordinal preferences (rankings of alternatives) in a way which is consistent with five plausible requirements. Using very similar axioms, Keeney [2] later showed that a consistent aggregation of cardinal preferences (utility values of alternatives measured on an interval scale) is possible and that a weighted sum of individual utilities provides a group utility function which fulfills all Arrow's requirements. The idea of an additive aggregation of cardinal preferences was introduced already earlier in the literature [3,4], and many researchers have extended and elaborated this concept later on [5–9].

However, providing cardinal evaluations of alternatives is a more demanding task for group members than providing just a ranking of alternatives. The benefits of consistent aggregation of

individual opinions therefore must be traded off against the higher cognitive burden that this approach places on group members.

The difficulties of providing exact cardinal values for the evaluation of alternatives, or at a more detailed level for some parameters of an underlying preference model (like weights to be assigned to different attributes) are well known in the area of decision analysis. To overcome these difficulties, many methods have been developed, which require only partial or incomplete information on a decision maker's preferences [10–19, e.g.]. Some of these methods, which are often labeled as disaggregation approaches [20,21] or robust ordinal regression approaches [22], actually use a ranking of alternatives as input from which they estimate parameters of a cardinal preference model. One could therefore view these methods as bridging the gap between cardinal and ordinal preference information.

Decision methods using incomplete information have also been proposed in the context of group decision making [22–27]. At the group level, incomplete information refers to the weights of group members in the aggregation procedure. At the individual level, incomplete information can either directly refer to the (cardinal) evaluation of alternatives by group members, or to some parameters of the underlying preference model (which in some methods is also an additive utility model).

In the present paper, we elaborate on the idea that decision methods using incomplete information cover a middle ground between cardinal and ordinal preference information. We study

* Corresponding author.

E-mail addresses: Rudolf.Vetschera@univie.ac.at (R. Vetschera), psarabando@estgv.ipv.pt (P. Sarabando), lmcdias@fe.uc.pt (L. Dias).

the effects of providing such information in different ways, which represent different points on the spectrum between cardinal and ordinal information. To provide a comprehensive analysis of the effects of different preference information, we consider a variety of outcome dimensions like conclusiveness of results or potential differences in the influence of group members. Since methods that rely on ordinal information (which is a boundary case in the spectrum of information levels we study) must necessarily violate at least one of Arrow's axioms, we also take such violations into account. In particular, we focus on the axiom of independence of irrelevant alternatives. Furthermore, we also study the impact of problem characteristics on our results. We analyze all these questions using a computational model.

The following section gives a brief overview of group decision methods using incomplete information and presents the specific approach on which our study is built. Section 3 defines the research questions in detail. Section 4 provides details about the computational study and the results are presented in Section 5. Section 6 discusses the results from the study and Section 7 offers some conclusions and suggests topics for further research.

2. Incomplete information in group decision making

2.1. A brief review

In the context of group decision aiding methods (see [27] for a review on negotiation support methods) there are different approaches to exploit the incomplete information provided by the decision makers (DMs): establishing robust (necessary) conclusions, assessing stability domains, and aggregating the information into a consensus result.

One class of approaches derives robust conclusions about the alternatives in the sense that these conclusions are verified for all the parameter values compatible with the incomplete information provided. These approaches usually look for preference relations among alternatives that necessarily occur. If one alternative is better than another one for all parameter vectors compatible with the incomplete information (possibly equally good for a subset of these vectors), then the latter alternative is said to be dominated. Such methods can also examine conclusions that hold for at least one parameter vector, namely to find out which alternatives may have the highest utility. If there exists any parameter vector compatible with the incomplete information such that an alternative has the highest utility, then this alternative is said to be potentially optimal. For example, Salo [6] finds dominance relations using mathematical programming approaches. Dias and Clímaco [24]'s framework proposes relaxing the concept of dominance considering a tolerance and a majority level. Greco et al. [22] use mathematical programming to identify necessary (dominance) or possible preference relations as consequences of indirect preference information provided by each DM, or agreed by the DMs.

A second class of approaches is based on assessing the domains of the parameter space that support some conclusions. These domains are considered as volumes of the parameter space (when such space is a polyhedron defined by linear constraints) or probabilities derived from a stochastic analysis (when parameter values are modeled as random variables having stochastic distributions). These interpretations coincide if the distributions are all uniform and independent. An early example is Bana e Costa [28], proposing the computation of an acceptability index combining individual preferences. Another example is SMAA, which uses Monte-Carlo simulation to compute for example the probability that an alternative occupies each position in the ranking of all alternatives; these probabilities are then aggregated into an indicator of the support for each alternative [23]. These

approaches can be combined with the first group [29] to provide more comprehensive information to decision makers.

A third class of approaches performs an aggregation of the incomplete information to directly derive a result (e.g., a ranking of the alternatives), or to yield a compromise vector of parameter values. As an example of the former (deriving a result directly), the interactive approach of Mateos et al. [25] uses Monte-Carlo simulation based on the incomplete information from the DMs to propose a ranking based on the mean utility of the alternatives. Another interactive approach, by Kim and Ahn [30], uses mathematical programming and net-flow aggregation to propose a ranking of the alternatives. Other methods yield a consensus parameter vector, which can later be used to obtain a ranking of the alternatives. This includes methods based on distances (e.g., [26]) and methods based on ordinal regression (e.g. [31]).

Let us note that it is possible to combine several of these approaches when dealing with a group decision or a negotiation situation [27].

2.2. Preference model and decision procedure

We consider a decision problem in which a group consisting of N_{mem} DMs has to rank a set A of N_{alt} alternatives or has to identify the best of these alternatives. We follow an additive aggregation approach based on e.g. Keeney and Kirkwood [4] and Dyer and Sarin [5]. In the case of complete information, the group utility of an alternative $A_i \in A$ can be written as

$$v(A_i) = \sum_{m=1}^{N_{mem}} w_m v_m(A_i) \quad (2.1)$$

where w_m is the importance weight of group member m , and $v_m(\cdot)$ is the value function of that DM. For a given and finite set of alternatives, there is only a finite set of values that each DM has to provide. To simplify the notation, we therefore denote the value which DM m assigns to the alternative A_i by v_m^i . We thus can rewrite (2.1) as

$$v(A_i) = \sum_{m=1}^{N_{mem}} w_m v_m^i. \quad (2.2)$$

We use notation $A_i >_m A_j$ to denote that A_i is preferred to A_j by a group member m (i.e., $v_m^i > v_m^j$), whereas $A_i > A_j$ denotes that A_i is preferred to A_j by the group.

Incomplete information can refer both to the weights w_m of the group members and to the individual evaluations v_m^i . In the present study, we consider both values to be uncertain. We follow a volume-based approach, which considers the uncertain parameters to be uniformly distributed across their respective domains, and use a Monte-Carlo method to sample parameter vectors which are compatible with the preference information available. Group members provide some information about preferences (e.g. a ranking of alternatives), and the method then generates values of the v_m^i in a fixed interval between zero and one which are compatible with this information. At the group level, we do not consider additional (problem specific) information on member weights (although such information could in principle also be accommodated), but we only consider a priori restrictions on these weights resulting from technical conditions like the scaling of weights, or variants of the non-dictatorship axiom.

For each parameter vector in the sample, the method calculates the group utilities of all alternatives. Since we follow a domain-based approach, we utilize the sampled group utilities to calculate two sets of indices. Both refer to probabilities and are approximated via the fraction of all the sampled parameter vectors which fulfill the corresponding conditions. Following the terminology of Kadzinski and Tervonen [29], these two indices are designated as

1. Pair-wise outranking indices p_{ij} , which indicate the probability (fraction of parameter vectors analyzed) that alternative A_i is preferred to alternative A_j at the group level.
2. Rank acceptability indices r_{ik} , which indicate the probability that alternative A_i obtains rank k in the group ranking.

These two sets of indices form the basic results of the group decision procedure under incomplete information which we study here. From them, further information can be derived to support the group in its decision process. For example, the probabilities r_{i1} indicate that an alternative A_i can obtain the best rank, so the set of alternative $\{A_i : r_{i1} > 0\}$ identifies the set of potentially optimal alternatives. Similarly, for each alternative the set of possible ranks that this alternative may obtain can be computed. If $p_{ij} = 1$ for some pair of alternatives A_i and A_j , the group considers alternative A_i to be better than A_j for all parameter vectors sampled and thus we can say that A_i (approximately) dominates A_j . Note that due to the finite sample size, it still cannot be said that no compatible parameters exist for which A_j would be considered to be better than A_i , thus dominance is only approximate (given the sample). Although it is possible to use optimization models to determine whether such parameters exist [29], this is not a topic of our research and we therefore limit our analysis to the simulation results.

3. Research questions

The main objective of this paper is to study the impact of different forms of incomplete information on the outcomes of the group decision procedure outlined in Section 2. We thus can formulate our main research question as follows:

How do different levels of incomplete information (on the values which group members assign to alternatives and on member weights) affect the outcomes at the group level?

To be more specific, we are interested in several dimensions of the group result. Since the procedure generates probabilistic information about preferences at the group level, a key issue is the conclusiveness of results. If the outcome of the procedure indicates that any alternative might be optimal with about the same probability, this result will not provide much support for the group in its decision making task. It is quite obvious that providing more precise inputs on the group member's preferences will lead to more conclusive results (e.g., fewer alternatives being identified as potentially optimal). The ranking of different information levels in terms of conclusiveness is therefore easily predictable. However, it is not only the ranking that is important here, but also the differences, i.e., how much does conclusiveness improve when information becomes richer. To perform this type of analysis, we not only study the cases of purely cardinal and purely ordinal information, but also an intermediate level, in which group members provide a ranking of the differences between alternatives in addition to a ranking of alternatives. By considering the differences in outcomes, we can determine whether this kind of information produces results which are closer to cardinal or to ordinal preference information.

Outcomes of group decisions, even if they are exact and not probabilistic, can still be evaluated according to several dimensions. Two obvious dimensions are fairness, i.e. how balanced the result is in reflecting the interests of different group members, and efficiency [32,33]. It is easy to show (as we will provide in the following section) that the proposed approach will only identify efficient alternatives as potentially optimal. By varying group members' weights, it allows for scanning the set of efficient alternatives, and we study how well this set can be explored using different types of information. With respect to fairness, different levels of incomplete information might have other effects. Cardinal

information about a group member's preferences implicitly gives more weight to group members who state that the difference between two alternatives is rather large for them compared to group members who indicate that the same alternatives for them are close to each other. By moving towards ordinal information, we expect the influence of group members on the final results to be more evenly distributed.

To summarize these considerations, we can therefore formulate our first research question more precisely as follows:

Research question 1 (RQ 1): What is the impact of different levels of information on group members' values of alternatives, and on the weights of group members, on outcomes, in particular:

- How much does the provision of information on differences in values improve the conclusiveness of results, compared to providing just rankings?
- How well can the set of efficient alternatives be scanned using different types of preference information and weights?
- What is the impact of providing different levels of information on the balance of influence by group members on the group outcomes?

Another aim of this study is to explore the limits of the proposed approach. Since Arrow's impossibility theorem [1] is a general result for any social choice function operating on ordinal preferences, it is obvious that the proposed approach must also violate at least one of Arrow's axioms. The critical issue here is Arrow's *Independence of Irrelevant Alternatives (IIA)* condition, which basically states that the ranking of any two alternatives by the group must not change depending on the availability of a third (the "irrelevant") alternative. If an alternative which is either the best or the worst in one member's ranking is dropped, this will affect the scaling of values of that member and consequently the outcomes of the procedure.

However, for actual applications of the procedure, the question is not whether such rank reversals can theoretically occur, but to which extent they will actually affect the results of the procedure. Thus, we will use our computational model also to answer the following research question.

Research question 2 (RQ 2): How frequently do violations of the IIA condition occur for different levels of information on group members' preferences and on weights?

Finally, results of computational studies always depend on the specific settings being analyzed. Apart from problem dimensions such as the number of alternatives and group members, which can easily be set for the simulations and for which we can study a wide range of possible values, more subtle characteristics of the problem might also have an impact on results. In particular, it might make a difference whether the group members already have very similar preferences about the alternatives in discussion, or whether there is strong disagreement among group members. Consider for example the extreme case in which all group members fully agree on the ranking of alternatives. In that case the group result will obviously always be the same ranking as that of all members, regardless of the kind of information provided, and all the effects to be analyzed in the previous two research questions will disappear. Therefore, we intend to study.

Research question 3 (RQ 3): What is the impact of problem characteristics, in particular, level of conflict, on the relationships postulated in RQ1 and RQ2?

4. Computational study

4.1. Model overview

To analyze our research questions, we performed a computational study in which we simulated the use of different types of

incomplete information in a group decision problem following the approach outlined in Section 2. In this study, we compared three levels of information on values of alternatives, as well as three levels of information on group member weights. Concerning values of alternatives, the following variants were used.

- *Full cardinal information (C)*: As a benchmark, we assumed that group members are able to specify the exact cardinal utility values for each alternative.
- *Ranking of differences (D)*: Here we assumed that group members provide a ranking of alternatives, as well as a ranking of differences of adjacent alternatives in their rankings.
- *Ranking of alternatives (R)*: As the weakest level of preference information, we assumed that each group member only provides a personal ranking of alternatives.

To a certain extent, the assumption of full cardinal information contradicts the main aim of decision models under incomplete information to simplify the cognitive task for the decision makers. We therefore view this setting mainly as a benchmark, against which the other two more realistic settings can be evaluated and which provides information on the loss of precision due to providing only incomplete information.

The three levels of information on member weights used were the following:

- *Equal weights (E)*: As a benchmark, we used a setting with equal weights for all group members. While it would be possible to simulate “true” unequal weights, such a setting would not provide additional insights. By such an approach, we could have studied the difference between arbitrary vectors and equal weights, but this question is not a topic of our study.
- *Non-dictator weights (N)*: As a second level, we considered all weight vectors $(w_1, \dots, w_{N_{mem}})$ which fulfill the condition $w_m \leq 0.5$ and thus the condition of independence of an imposed winner as defined in Dias and Sarabando [34]. Furthermore, we considered $\sum_{m=1}^{N_{mem}} w_m = 1$ and $w_m \geq 0 \forall m$.
- *General weights (G)*: As the third level, we considered arbitrary weight vectors fulfilling the conditions $\sum_{m=1}^{N_{mem}} w_m = 1$ and $w_m \geq 0 \forall m$.

In total, we thus considered $3 \times 3 = 9$ different information settings. In the following, we will denote these information settings by two letter codes representing the information level on values and on member weights as indicated in the above lists. Thus, for example “DN” refers to a setting on which a ranking of differences (D) is specified for values, and non-dictatorship weights (N) are used for the group members.

Fig. 1 provides an overview of the simulation framework. For each simulation experiment, a random problem instance was generated by drawing $N_{alt} \times N_{mem}$ random values (one for each of the N_{alt} alternatives and each of the N_{mem} group members) from a uniform distribution. These values were subsequently rescaled, so that the best alternative in each member’s ranking received a value of one, and the worst alternative a value of zero. For this study, we thus considered evaluations of group members to be completely independent of each other. This could, for example, reflect a setting in which group members represent different interests and thus evaluate alternatives according to different (and uncorrelated) criteria.

Given these “true” cardinal values, the simulation program then calculated the ranking of alternatives, and the ranking of differences between adjacent alternatives for each member. These rankings were used as an input to the analysis as outlined in Section 2. In the case of true cardinal values, these values were directly used. For information level “R” (rankings), the procedure generated N_{alt} random values for each group member, which were

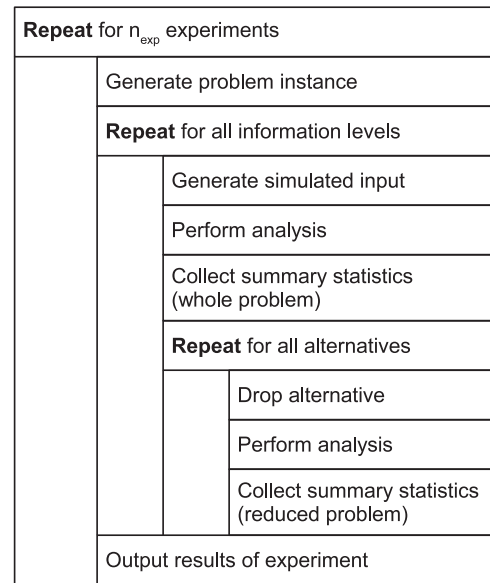


Fig. 1. Simulation framework.

Table 1
Simulation parameter settings.

Parameter	Values used
Number of group members	3, 5, 7, 9
Number of alternatives	5, 10, 15, 20
Problem instances generated	2000
Parameter vectors for each problem instance	10,000

subsequently rescaled to the zero-one interval, sorted, and assigned to alternatives according to the group member’s ranking. For information level “D” (differences of values), we used a variant of the method of Butler et al. [35], as described in [19]. Member weights for the general setting (type “G”) were also generated using the method of Butler et al. [35]. For the non-dictator weights, we used a rejection method and generated weight vectors until one was found which did not contain any weight larger than 0.5.

One of our research questions addresses the IIA property. For this analysis, we subsequently dropped each alternative from the original problem and repeated the analysis for each reduced problem. In cases in which the best or worst alternative of a group member is dropped, this will lead to a different scaling of the generated random values assigned to alternatives, which could generate violations of the IIA axiom.

To test the sensitivity of our results for different parameter settings, we performed experiments for different problem dimensions. Table 1 summarizes the main parameter setting used for this study.

In the following Section 5, we will only present results referring to the extreme parameter values (shown in boldface in Table 1). The remaining results lie in between those extreme results, and can be obtained from the authors.

The simulation program was implemented in Object Pascal using the open source Free Pascal compiler (www.freepascal.org). The source code can be obtained from the authors upon request.

4.2. Measurement of outcome variables

As explained in Section 2.2, the group decision procedure generates pairwise outranking indices p_{ij} and rank acceptability indices r_{ik} , which provide probabilistic information on the group

ranking. From this data, we derive measures to analyze our research questions. The first research question deals with the impact of different information levels on the efficiency, fairness, and conclusiveness of results. So we need to measure these three concepts.

We consider the outcome of a simulation to be conclusive if it strongly reduces the number of alternatives that should be considered by the group. To measure conclusiveness, we therefore consider the number of potentially optimal alternatives i.e. alternatives for which $r_{i1} > 0$.

Even in the case of information level “R” (only ranks of alternatives are provided), inefficient alternatives cannot be potentially optimal. An alternative A_i is inefficient if there exists an alternative A_j such that A_j is preferred over A_i by all group members. Thus any randomly generated set of values will assign a higher value to A_j for each group member than to alternative A_i , yielding necessarily a higher group utility and the result that $p_{ij} = 0$. Since only the efficient alternatives can be potentially optimal, the main information that can be derived is therefore how much of the efficient set can actually be explored by using random weights and/or values. It should be noted that neither having one optimal alternative nor covering the entire set of efficient alternatives is necessarily the best result; both types of results provide valuable information to the DMs. One interesting aspect in this context is whether the intermediate forms of incomplete information (non-dictator weights and information on differences) are closer to the case of exact values, or to the case of most incomplete information we consider here.

Apart from efficiency, we also consider fairness. A group ranking can be considered to be balanced if the rankings of individual members are equally well represented in the group ranking. For an exact group ranking, one could therefore use some measure of correlation between group and individual rankings. However, the pairwise outranking indices provide only probabilistic information. To measure the correspondence between this group level information and the ranking of member m , we define the following index of correspondence with the group ranking:

$$GC_m = \frac{\sum_{i=1}^{N_{alt}} \sum_{j:A_i \succ_m A_j} p_{ij}}{(N_{alt}(N_{alt}-1))/2} \quad (4.3)$$

where $A_i \succ_m A_j$ indicates that member m prefers A_i over A_j . Note that we use the sum of probabilities, so GC_m is the average probability that for any pair of alternatives, the ranking of that pair by the group corresponds to the member's ranking. Assuming that all p_{ij} are independent of each other (which is obviously not the case, since for a given parameter vector the group ranking is transitive), one could calculate the probability that the group ranking fully agrees with the member's ranking as the product of these probabilities. However, the product would be zero in case that the group never agrees with the member on just two alternatives, so we consider the average defined in (4.3) as the more robust measure. A value of $GC_m = 1$ would mean that for all parameter vectors, the group always agrees with the member on the ranking on any two alternatives, indicating that the member has a very strong influence on the group. Thus, we can consider the influence of the most influential member as an indicator of how balanced the group outcome is with respect to the individual members' opinions.

However, GC_m has still one major drawback, since it does not account for the level of agreement between the rankings of group members. If all members initially agree on the ranking of all alternatives, we would obtain $CG_m = 1$ for all members. Still we should not conclude that the most influential member (which in that case would be any member) is a dictator who always

determines the group outcome. To correct for this effect, we propose to standardize GC_m by the initial level of correspondence between group members, which can be defined analogously to (4.3) as

$$MC_m = \frac{\sum_{n \neq m} |\{(i,j) : A_i \succ_m A_j \wedge A_i \succ_n A_j\}|}{N_{alt} \cdot (N_{alt} - 1) \cdot (N_{mem} - 1)/2} \quad (4.4)$$

which for each member counts the number of other members who have the same preference on any pair of alternatives, standardized by the number of other members ($N_{mem} - 1$) and the number of pairs of alternatives $N_{alt} \cdot (N_{alt} - 1)/2$. Thus MC_m is the average fraction of other members who agree with member m on the ranking of any pair of alternatives. From these two values, we can derive a measure of member m 's influence as

$$IF_m = \frac{GC_m}{MC_m} \quad (4.5)$$

In the spirit of “non-dictatorship”, we consider the highest influence factor $\max_m IF_m$ as an indicator of how (un-)balanced the impact of members on the group decision is. As alternative measures of the dispersion of influence factors, we also calculated the range of IF_m (i.e. $\max_m IF_m - \min_m IF_m$), and its standard deviation. In Section 5, we only present results on $\max_m IF_m$, the other results are provided in the online supplement to this paper.

The second research question addresses possible violations of the IIA axiom. Here we have again a similar problem. Since the analysis only yields probabilistic information, we first have to define what a rank reversal actually means in this context. For this purpose, we consider the median rank of each alternative, and we define a rank reversal to occur if in the presence of alternative A_k , the median rank of alternative A_i is larger than that of alternative A_j , while in the absence of alternative A_k , the median rank of A_i is smaller than that of A_j .

We use the median rather than the mean rank for this analysis, as the median is a more robust measure of location. Since median ranks are integer values (or in the case of ties values halfway between two integers), the median also provides a more clear-cut definition of a rank reversal. Using the mean, two alternatives could have fractional mean ranks which are very close to each other, and which could be easily changed to two very close values in the opposite order. Nevertheless, we conducted a similar analysis also for mean ranks, the corresponding results are available in the online supplement to this paper.

If each alternative is dropped in turn, the maximum number of rank reversals that can occur during such experiments is $N_{alt}(N_{alt}-1)(N_{alt}-2)/2$, so the actual number of rank reversals is standardized by this factor to obtain comparable results.

Our last research question refers to the influence of the initial level of conflict (or agreement) on outcomes. This can conveniently be measured by the average value of MC_m across group members.

5. Results

5.1. RQ1: impact of information levels

Our first research question concerned the impact of different information levels on the conclusiveness of the results, as well as on the efficiency and fairness of group results.

A main result of the incomplete information model is the set of potentially optimal alternatives. Table 2 shows the average number of potentially optimal alternatives for the four extreme parameter settings studied. Obviously, for exact parameters, only one alternative can be optimal (for randomly generated problems the

probability of a tie is negligible), and this number increases the less precise the information becomes.

Although Table 2 shows the absolute numbers of potentially optimal alternatives in each of the four scenarios, the main interpretation of the table lies in the comparison of different information types within each problem size. Differences between problem sizes depend on specific characteristics of each problem size and should not be over-interpreted. In particular, the simulation did not eliminate dominated alternatives from the problems, since we considered the existence of dominated alternatives a realistic scenario. Given that members do not know each other's ranking, they might not even be aware that one alternative is dominated (in terms of their individual rankings). Obviously, it is more likely that an alternative is dominated in the case of only three group members (compared to nine members), this partially explains why numbers for the larger groups are higher. In fact, in our simulation data, for three members on average only 66.21% and 36.21% out of 5 and 20 alternatives respectively were efficient, whereas the number of efficient alternatives is close to 100% for nine members.

Table 2
Average number of potentially optimal alternatives.

Values	Weights			All	Weights			All
	Equal	Non-dict	General		Equal	Non-dict	General	
3 members 5 alternatives								
Card.	1.000	2.168	2.915	2.027	1.000	4.630	4.673	3.434
Diff.	1.837	2.801	3.138	2.592	1.999	4.718	4.756	3.824
Ranks	2.622	3.276	3.310	3.069	3.476	4.889	4.894	4.419
All	1.820	2.748	3.121	2.563	2.158	4.745	4.774	3.893
3 members 20 alternatives								
Card.	1.000	3.101	5.313	3.138	1.000	13.716	14.066	9.594
Diff.	1.731	3.757	5.654	3.714	1.807	13.898	14.214	9.973
Ranks	4.193	6.109	6.877	5.726	4.563	15.444	15.661	11.889
All	2.308	4.322	5.948	4.193	2.457	14.352	14.647	10.485

Comparing different information levels relative to each other, it is clear from Table 2 that uncertainty about member weights has a strong effect on outcomes. In the case of nine members, a large fraction of alternatives can become optimal, even if member weights are restricted to be less than 0.5. This restriction does not have a strong effect in the case of nine members, since the probability that any member will have such a large weight is very small. In the simulations, on average only 1.0364 weight vectors had to be generated for every weight vector fulfilling the non-dictatorship condition in problems with nine members. In comparison, the ratio was 3.9998 for problems with three members. Consequently, results for the two cases of uncertain member weights are very similar for nine members.

Concerning the effect of uncertainty in values, providing information on differences in values rather than just a ranking has a strong effect. The set of potentially optimal alternatives in that case is almost as small as in the case of exact values. A nonparametric Wilcoxon test indicates that the difference between the number obtained for exact values and a ranking of differences is always significantly smaller than the difference between the results obtained from a ranking of differences and a ranking of alternatives. Information on weights and utility values cannot directly be compared, since a ranking of alternatives still provides some information, while in the case of member weights the extreme scenario (general weights) does not provide any information.

The set of potentially optimal alternatives also provides some insight about efficiency of results. As we have already shown, an alternative which is not Pareto optimal must have a probability of zero of being the optimal alternative. Therefore, we only analyze which fraction of efficient alternatives is identified as potentially optimal.

Fig. 2 provides an analysis relating potential optimality to the actual efficiency of alternatives. It shows the distribution of the ratio

$$\frac{\text{number of potentially optimal alternatives}}{\text{number of efficient alternatives}} \tag{5.6}$$

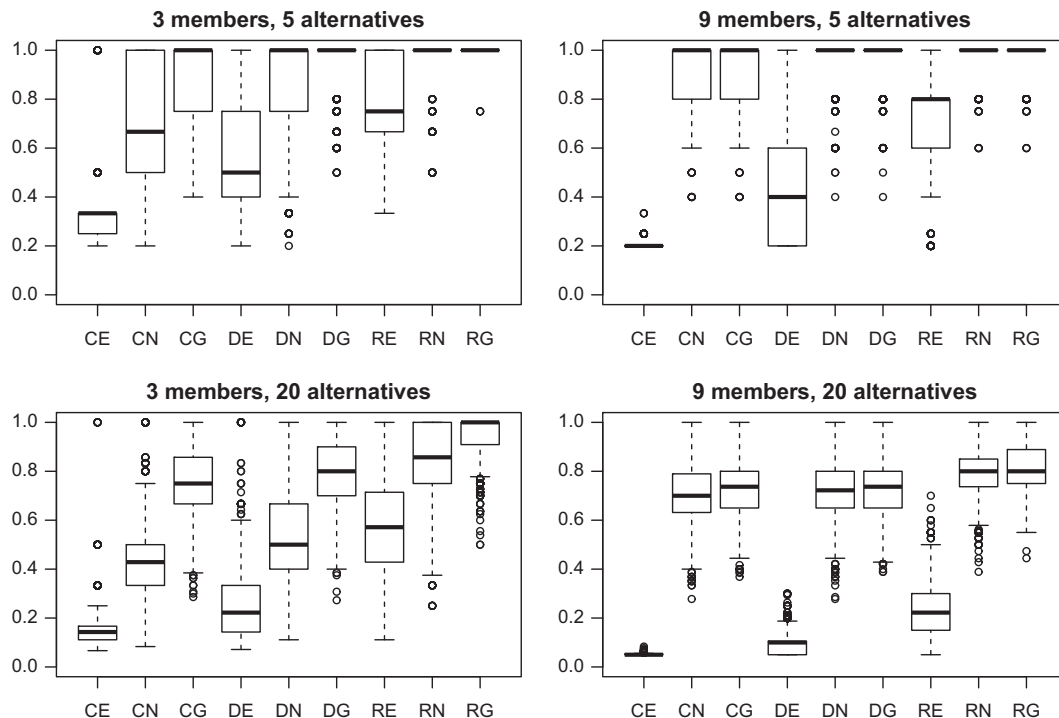


Fig. 2. Fraction of efficient alternatives identified as potentially optimal.

for each problem setting and information level. It should be noted that in some problems, this measure cannot reach a value of one. Alternatives, which are efficient, but which are dominated by some linear combination of other alternatives will not be optimal for any weight vector, and thus will not be potentially optimal. This is more likely to happen in the case of 20 alternatives, thus the values shown in Fig. 2 are smaller for these problems. Although in the case of exact values and equal weights, there is only one potentially optimal alternative, the fraction in that case varies, since the denominator of (5.6) is different between problems.

Again, the distinction between non-dictator weights and general weights becomes irrelevant for settings with nine group members. In small problems (with five alternatives) almost all efficient alternatives can become optimal for at least some parameter vectors in the more uncertain settings. In contrast to the previous results, providing a ranking of differences has not such a strong effect, but rather is close to the middle between exact values and providing only rankings.

As indicated in Section 4.2, we measure fairness of outcomes by the relative influence of member rankings on the group results. Table 3 shows the averages of maximum influence factors as defined in Eq. (4.5). Although some of the values are quite similar, a Wilcoxon test indicates that results for all types of value information, as well as for all types of weight information, are significantly different from each other at the $p < 0.1\%$ confidence level. As we expected, providing less information about preferences leads to a more equal influence of group members. Increasing the number of alternatives decreases the maximum influence of members, which is plausible since the group would then have to agree with the member on the ranking of a larger number of alternatives. Likewise, in larger groups each member has only a smaller influence. In the large groups, the difference between non-dictator weights and general weights is again very small (although it is still statistically significant here). We also find again that providing information about differences in values leads to results which are more similar to the specification of exact values than to providing just ranking information.

5.2. RQ2: independence of irrelevant alternatives

Our second research question concerned the extent to which the approach is affected by violations of the IIA axiom. Fig. 3 shows the fraction of possible reversals of median ranks that was observed in the experiments. Rank reversals do occur at a rate of about one quarter of the possible maximum number. However, several facts should be noted in order to put this number in perspective: in our simulations, we systematically dropped each

alternative and checked whether eliminating this one alternative leads to a rank reversal. By construction of the method, a scale change that leads to a rank reversal can happen only if the alternative dropped is the best or worst one for some group member (which would lead to a different scaling of the utility values). Thus, in practice, no rank reversal occurs if some other alternatives are dropped. Furthermore, we considered all rank reversals along the entire preference relation. In practical applications, a reversal between for example the 17th and the 18th out of 20 alternatives will probably not cause a problem, only reversals among the first few alternatives might lead to a different decision. Finally, we performed a new simulation for every reduced problem, which also meant generating new random parameter vectors. Although the sample size was large enough that different random streams on average make no difference, in some cases this might have slightly affected the outcome, and if it did, it would have inflated the number of rank reversals.

As could be expected, increasing uncertainty about values also increases the occurrence of rank reversals. In that case, providing incomplete information in the form of differences leads to almost the same results as providing it just in the form of a ranking of alternatives. An interesting phenomenon occurs with respect to member weights: in some cases (in particular for the large problems of nine members and 20 alternatives) equal (and thus fixed) weights lead to *more* rank reversals than incomplete information about weights. For problems with nine members, the difference between non-dictator and general weights is again negligible, however, for problems with only three members the increased uncertainty of general weights is also reflected in a slightly higher rate of rank reversals.

5.3. RQ3: impact of conflict level

Our final research question deals with the impact of problem characteristics, in particular the level of conflict among group members, on the outcome dimensions. We measure conflict indirectly, via the initial agreement between the rankings of group members. As could be expected, if group members' rankings are in more agreement initially, this leads to higher conclusiveness of results, as can be observed from the scatterplots for different information levels shown in Fig. 4 for the most complex problem. To test this relationship statistically, we performed separate regression analyses between the agreement level and the number of potentially optimal alternatives for all parameter settings and information levels. To make results comparable across parameter settings, we divided the number of potentially optimal alternatives by the total number of alternatives for this analysis. The results summarized in Table 4 confirm that the negative relationship between these two variables is a robust phenomenon for all the parameter settings we analyzed. The regression coefficients of conflict level are always negative, indicating that the negative relationship which Fig. 4 illustrates for the case of nine members and 20 alternatives also holds for the other problem sizes.

A similar effect can be observed concerning the fairness of outcomes, as indicated in Table 5. Most regression coefficients are again negative, indicating that higher levels of initial agreement have an equalizing effect on outcomes. This is not surprising, since the initial level of agreement formed the denominator of our measure of influence. However, for the most complex problem, this relationship is reversed (although in some cases it is only weakly significant). In particular, when member weights are randomly generated, a higher initial agreement leads to a higher influence factor, which means that the numerator in IF_m , i.e. the correspondence between group rankings and the member's ranking, increases even faster than the denominator.

Table 3
Average influence factors.

Values	Weights			All	Weights			All
	Equal	Non-dict	General		Equal	Non-dict	General	
	3 members 5 alternatives				9 members 5 alternatives			
Card.	1.666	1.611	1.605	1.627	1.598	1.337	1.333	1.423
Diff.	1.628	1.599	1.599	1.609	1.550	1.328	1.324	1.401
Ranks	1.537	1.537	1.571	1.548	1.472	1.287	1.283	1.347
All	1.610	1.582	1.592	1.595	1.540	1.317	1.313	1.390
	3 members 20 alternatives				9 members 20 alternatives			
Card.	1.463	1.436	1.413	1.437	1.358	1.241	1.239	1.279
Diff.	1.458	1.434	1.412	1.435	1.354	1.240	1.238	1.277
Ranks	1.426	1.408	1.401	1.412	1.339	1.231	1.228	1.266
All	1.449	1.426	1.409	1.428	1.350	1.237	1.235	1.274

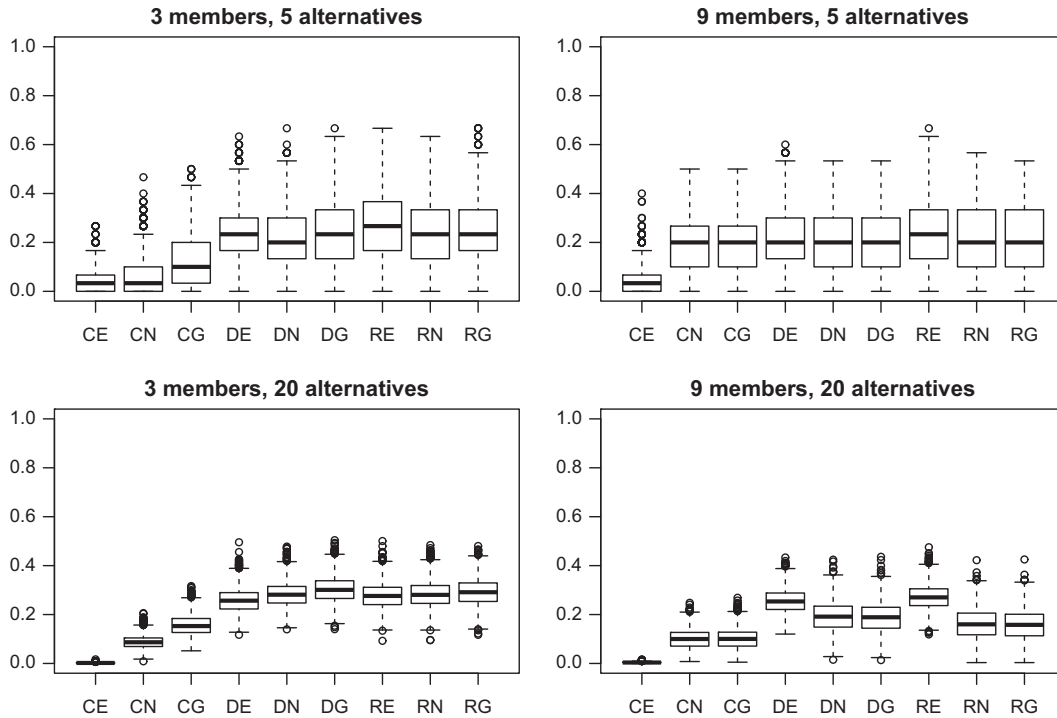


Fig. 3. Fraction of possible violations of IIA that occurred.

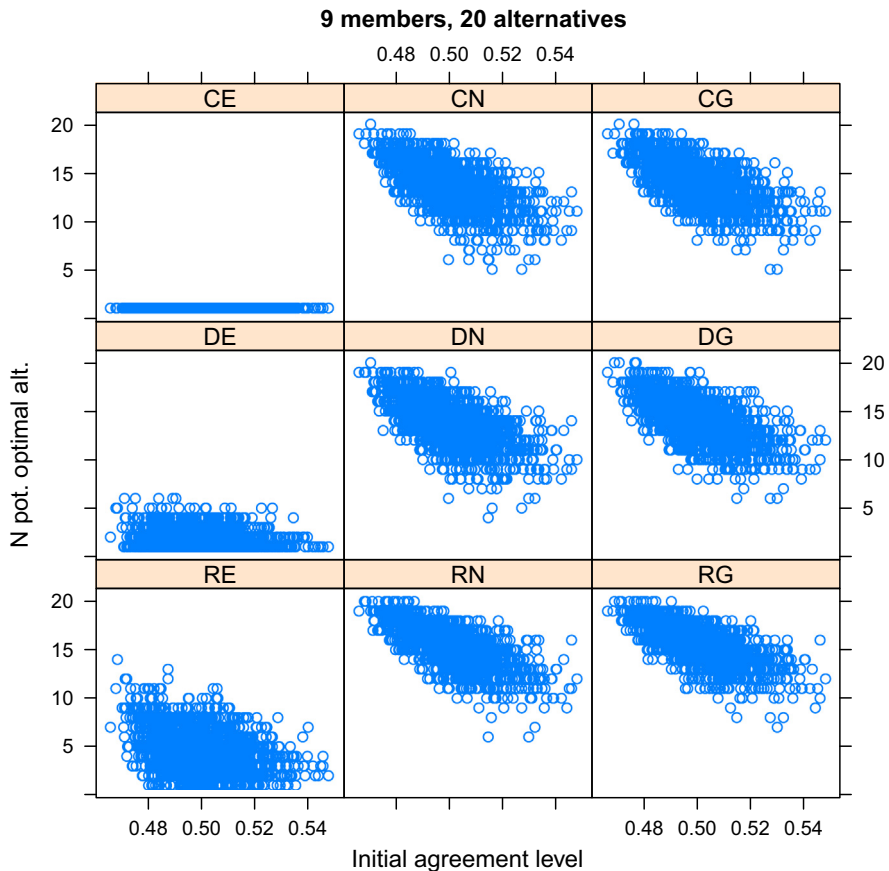


Fig. 4. Number of potentially optimal alternatives vs. initial agreement for different information levels.

A similar picture is obtained for the effect of initial conflict on rank reversals (Table 6). The regression analysis indicates a significantly positive effect for large problems, indicating that higher levels of agreement will lead to a situation in which more

rank reversals occur. However, as Fig. 5 shows, the actual number of agreement reversals does not increase much across the range of agreement levels contained in our data, the comparatively high coefficient results from the small changes in agreement levels. The

Table 4
Regression coefficients between initial agreement levels and number of potentially optimal alternatives.

Values	Weights			Weights		
	Equal	Non-dict	General	Equal	Non-dict	General
	3 members 5 alternatives			9 members 5 alternatives		
Card	0.0000	-0.7091***	-0.9622***	0.0000	-2.1035***	-1.8876***
Diff	-0.4299***	-1.0977***	-1.2275***	-2.0058***	-1.9056***	-1.6747***
Ranks	-0.8795***	-1.4514***	-1.4560***	-4.2120***	-1.1823***	-1.1314***
	3 members 20 alternatives			9 members 20 alternatives		
Card	0.0000	-0.3716***	-0.6428***	0.0000	-5.4693***	-5.2004***
Diff	-0.1003***	-0.5247***	-0.7529***	-0.5013***	-5.4268***	-5.2781***
Ranks	-0.6565***	-1.0854***	-1.1659***	-2.7831***	-5.4776***	-5.1086***

* $p < 5\%$, ** $p < 1\%$, *** $p < 0.1\%$.

Table 5
Regression coefficients between initial agreement levels and influence factors.

Values	Weights			Weights		
	Equal	Non-dict	General	Equal	Non-dict	General
	3 members 5 alternatives			9 members 5 alternatives		
Card	-2.2959***	-2.2102***	-2.3996***	-2.4099***	-0.5140***	-0.5056***
Diff	-2.1996***	-2.1870***	-2.4045***	-2.0125***	-0.4205***	-0.4167***
Ranks	-1.8864***	-2.0200***	-2.3803***	-1.3117***	-0.0715***	-0.0729***
	3 members 20 alternatives			9 members 20 alternatives		
Card	-1.7115***	-1.5840***	-1.5231***	-1.1791***	0.0886*	0.0917*
Diff	-1.6925***	-1.5700***	-1.5209***	-1.1583***	0.0908*	0.0897*
Ranks	-1.5278***	-1.4207***	-1.4839***	-0.9382***	0.2078***	0.2057***

* $p < 5\%$, ** $p < 1\%$, *** $p < 0.1\%$.

Table 6
Regression coefficients between initial agreement levels and rank reversals.

Values	Weights			Weights		
	Equal	Non-dict	General	Equal	Non-dict	General
	3 members 5 alternatives			9 members 5 alternatives		
Card	-0.1092***	-0.2686***	-0.3991***	-0.4743***	-1.2012***	-1.1276***
Diff	0.0027	0.0156	0.0178	-0.0136	-0.5270***	-0.5750***
Ranks	-0.0758**	0.0408	0.0151	0.1201	-0.7545***	-0.7675***
	3 members 20 alternatives			9 members 20 alternatives		
Card	-0.0092***	-0.2769***	-0.3364***	-0.0454***	0.2999***	0.3749***
Diff	-0.0676**	-0.0782**	-0.0085	-0.1728*	1.7379***	1.7281***
Ranks	0.0114	0.1095***	0.1281***	0.0768	2.1069***	2.0556***

* $p < 5\%$, ** $p < 1\%$, *** $p < 0.1\%$.

graph also suggests that the relationship might not be linear for higher levels of agreement.

6. Discussion

The main research question of this paper was to study the effects of providing different types of incomplete information in group decision models. Our results show that providing information on the ranking of differences, rather than just on the ranking of alternatives, has a quite strong effect on outcomes and in some cases even brings the results close to those obtained under complete information. One unexpected result of our simulations

is how much the impact varies between different outcome dimensions. It could be expected that the effect of providing difference information is moderated by problem characteristics such as the number of alternatives. If group members specify their ranking of many alternatives, as well as their ranking of difference between them, this provides a lot of information on their preferences which can be expected to lead to a clear ranking at the group level and thus significantly reduce the number of potentially optimal alternatives. However, one would a priori expect that this additional information affects all outcomes in roughly the same way. If additional information restricts the possible number of different rankings at the group level, this should also be reflected in the other outcome dimensions. While this is the case to some

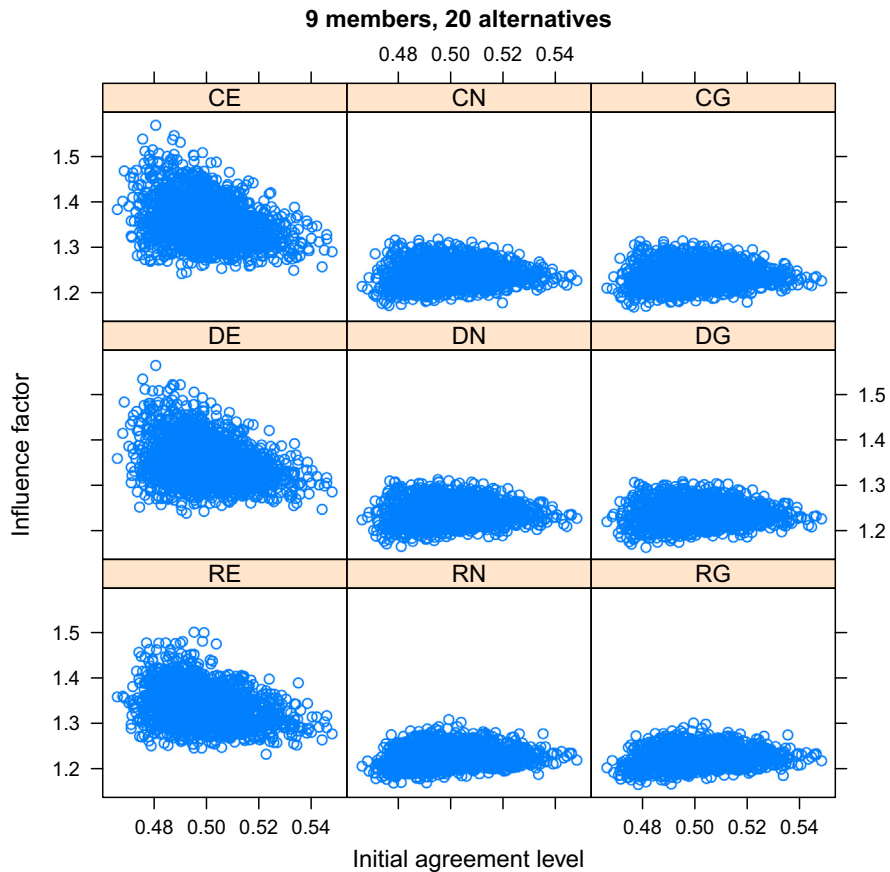


Fig. 5. Number of rank reversals vs. initial agreement for different information levels.

extent, our results show that the magnitude of these effects varies considerably across outcome dimensions.

Nevertheless, for many outcomes and in particular for the number of potentially optimal alternatives, our results clearly indicate the benefits of providing preference information in the form of a ranking of differences between adjacent alternatives. Of course, these benefits must be evaluated against the possible disadvantages of providing such information. Any additional preference information that is required from group members also increases the cognitive load on them. This additional burden is in conflict with the main goal of decision models under incomplete information, namely to provide easily usable tools to decision makers.

The net benefits of providing information on a ranking of differences therefore depend on whether it is possible to develop intuitive methods for providing such information. One possibility is a graphical representation in the form of a slider, on which group members can position alternatives to indicate whether they consider neighboring alternatives to be close to each other, or far apart. In fact, such a slider could also be used to provide cardinal evaluations of alternatives. Interpreting positions on a slider as exact cardinal values, however, require users to provide a much more precise evaluation of alternatives than if they only need to consider whether differences are bigger or smaller. Decision makers might be unable, or unwilling, to provide such precise information and will probably feel more comfortable if they know that only the ranking of differences will actually be used in the procedure. Alternatively, pairwise comparisons about the difference in neighboring alternatives could be elicited directly to obtain the required information.

As we have seen, moving towards cardinal information about alternatives not only has an impact on the conclusiveness of

results, but also makes the influence of group members less balanced. One could argue that such unequal influence is still fair, and if one group member has really strong feelings about the ranking of two alternatives, this group member's opinion should have a stronger influence than the opinion of a member to which the two alternatives are almost the same. However, this effect also creates an incentive for manipulation: if a member can increase his or her influence by claiming that the difference between two alternatives is large, then one might try to do this in order to gain more influence. This possibility for manipulation is limited by the fact that all alternatives must be located within a fixed scale interval, and thus claiming a large difference between two alternatives implies a smaller difference between other alternatives. However, this instrument could still be used strategically, in particular if one has some information about the preferences of other members. A member could then assign large differences to those pairs of alternatives about which others have a different opinion.

These two phenomena together represent an interesting trade-off in the design of group decision methods: eliciting more precise (closer to cardinal) preference information from group members on the one hand increases the conclusiveness of results, but on the other hand provides more incentives and more opportunity for manipulation. A better understanding of this trade-off makes it possible to select the level of information to be used in a particular situation, depending on the characteristics of the situation.

In addition to the different levels of information on preferences, we also studied the effect of different types of weights. Our results here show that for moderate to large group sizes, the non-dictatorship condition has only a very weak effect. Even if weight vectors are created randomly, a group member would only very rarely receive a weight which is large enough to make him or her a dictator.

While the non-dictatorship condition thus does not seem to pose a problem for group decision models using incomplete information, the IIA condition is more critical. Our results indicate that IIA violations occur at a rate which cannot be ignored. As we have already outlined in the previous section, our simulation might somewhat exaggerate this possibility compared to those cases of rank reversals which are really relevant in practical applications. Still, this issue needs to be taken into account. One consequence that we can draw for practical applications is that it will be necessary to carefully analyze the set of alternatives at the beginning of the decision process to avoid that alternatives need to be added later. The problem of removing an alternative is less important, because the group can still keep that alternative as an artificial option that cannot be chosen but that was useful to elicit information.

Our final results about the impact of conflict levels indicate that problem characteristics do have an influence on the phenomena we studied. This is of course a general limitation of computational models that they will only allow conclusions for the specific setting being analyzed. However, our analysis of the impact of different conflict levels has at least identified one factor that has an impact on the results of such procedures, as well as some insights in the direction and size of its effects.

7. Conclusions and future research

As the discussion in the previous section has shown, our experiments have provided several insights into the effects of providing different information levels in group decision models under incomplete information. However, our study also has several limitations which need to be taken into account in future research, and in this section, we will outline possible strategies to overcome those limitations.

One area which needs attention in future studies is the refinement of outcome measures. In particular, we have considered violations of the IIA axiom in a very straightforward manner in this study. From an axiomatic point of view, any rank reversal that occurs is a violation of the axiom, and thus all rank reversals are equal. From a more application-oriented perspective, however, rank reversals are of different importance. Rank reversals between top-ranked alternatives will have an impact on actual decisions being made by the group, while reversals of alternatives which would not be selected anyway are not that important. Although any weighting or classification of rank reversals is to a certain degree arbitrary, it still might be useful to consider only reversals up to a certain rank, or give reversals among top-ranked alternatives a higher weight in calculating some severity score of rank reversals in future studies.

Concerning the inputs to the procedure, we assumed that group members would individually rank the alternatives without ties. It would be possible to extend the analysis to cases in which the group members may consider two alternatives to have the same rank.

Another limitation of our study is that we assumed that members provide preference information on all alternatives. In a more general setting preference information could be provided on a subset of the alternatives to elicit individual preferences that might then be applied to other alternatives beyond the sample set. However, for problems involving a limited number of alternatives, as we are using in our simulations, one can expect the DMs are interested in comparing all alternatives explicitly to make their preferences clearer. We can also note that to consider full cardinal information as an extreme benchmark case necessarily involves all alternatives. Using only a subset of alternatives for the other cases

would confound effects of leaving out some alternatives with the effect of providing different types of information.

Our results also indicate that contextual factors like the level of conflict among group members have an impact on the information effects we are studying. Level of conflict by itself is only one measure of context, which could be further refined. In particular, a more realistic model should not just be based on the level of conflict, but on those factors which influence that level. Our model could thus be enhanced by explicitly modeling the decision processes of members that lead to their (cardinal) evaluations of alternatives. For example, one could model the underlying decision problem as a multi-criteria problem. Many different levels of detail are possible in such a model. One could directly assume that group members have certain (and different) partial values for each alternative in each attribute. Alternatively, one could base these values on some objective properties of the alternatives, which would be the same for all group members, but to which group members apply different marginal value functions. If one considers objective attribute values of alternatives, the correlation between these attributes might become another characteristic of the problem that has to be taken into account.

From a formal perspective, the model we used in this study could also be interpreted as the model of one single decision maker, who provides incomplete information at different scale levels on the evaluation of alternatives in several attributes. However, this analogy is rather limited, since an individual decision maker could also specify some (incomplete) information on the weights of attributes, and a non-dictatorship condition would not be appropriate in that setting.

In our view, the relationship of the present model to multi-criteria decision making should thus be exploited by developing integrated models, which consider both the individual-multi-criteria decision process of group members (possibly also involving incomplete information) and the aggregation at the group level as two distinct, but linked steps. Developing such deeper and more realistic models of group decision problems opens a wide area of possible extensions to our model that would probably make our results more applicable to real life situations. Of course, ultimately, these phenomena should be studied in the context of actual decision problems rather than for randomly generated data. Even within these limitations, our study offers some first insights into the effects of information on group decision procedures under incomplete information.

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Appendix A. Supplementary data

Supplementary data associated with this article can be found in the online version at <http://dx.doi.org/10.1016/j.cor.2014.05.021>.

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