

A MARKOV CHAIN MODEL IN DECISION MAKING

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Abstract

An absorbing Markov chain is introduced in order to give a mathematical formulation of the decision making process. Examples are also given to illustrate our results .

1. The decision making process

Decision making (d-m) is the process of choosing a solution between two or more alternatives , aiming to achieve the best possible result for a given problem.

It is obvious that this process has sense if, and only if , there exist more than one feasible solutions , together with the suitable criterion (-ia) that helps the decision maker to choose the best among these solutions .

A proposed solution is characterized as a feasible one if it satisfies all the natural restrictions imposed (onto the problem) by the real system that we study (e.g. if x denotes

the desired quantity of stock of a product, it must be $x \geq 0$).

From the other hand, the adoption of the suitable criterion , especially when , as it frequently happens, the results of a decision are affected by random events , depends upon the decision maker's way of thinking (e.g. optimistic or conservative criterion etc).

It is well known that the complexity of the problems of the everyday life led , from the beginning of the 1950's, to the development of a systematic methodology for d-m, based on Probability Theory, Statistics, Economics, Psychology etc and called **Statistical Decision Theory** (e.g. see [1]).

The d-m process involves the following steps :

d_1 = Analysis of the decision-problem (i.e. the problem on which one has to make a decision):
Understanding, simplifying and reformulating the problem ,

d_2 = Collection and interpretation of all the necessary information ,

d_3 = Determination of all the alternative feasible solutions, and

d_4 = Choice of the best solution (using the suitable criterion).

One could add one more step to the process: The verification of the chosen solution according to the results obtained by applying it in practice . But this step is extended to areas which, because of their depth and their importance for the administrative rationalism, have been autonomous and are examined separately .

Notice that the three first steps above are continuous . This means that the completion of each step does not happen very quickly, but it needs some time, during which the decision maker's reasoning is characterized by transitions between hierarchically neighbouring steps. The "flow-diagram" of the d-m process is represented in figure 1 below:

$d_1 \leftrightarrow d_2 \leftrightarrow d_3 \rightarrow d_4$ Fig. 1

2. The Markov chain model .

A Markov chain is a mathematical model for describing a certain type of stochastic process that moves in a sequence of phases through a set of states .

In earlier papers we have used Markov chains in problem solving ([3] and [5]), in order to describe the processes of modelling [4] and learning [6] etc .

We consider the reader familiar with the basic theory of finite Markov chains (for a good exposition see [2]) and especially of the absorbing chains ([2] , Chapt. III) .

Here, in order to give a mathematical formulation of the d-m process, we introduce a finite absorbing Markov chain having as states the steps of this process described in the previous section .

It is supposed that d_1 is always the starting state , while d_4 is the unique absorbing state , because , when the chain reaches d_4 , it is impossible to leave it .

Let us denote by p_{ij} the transition probability from state d_i to d_j , $i,j = 1,2,3,4$. Then the

transition matrix of the chain is

$$A = \begin{matrix} & \begin{matrix} d_1 & d_2 & d_3 & d_4 \end{matrix} \\ \begin{matrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ p_{21} & 0 & p_{23} & 0 \\ 0 & p_{32} & 0 & p_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

with $p_{21} + p_{23} = p_{32} + p_{34} = 1$ (1)

Let us also denote by $\ddot{o}_0, \ddot{o}_1, \ddot{o}_2, \dots$ the several phases of the chain and let

$P_i = [p_1^{(i)} \ p_2^{(i)} \ p_3^{(i)} \ p_4^{(i)}]$ be the row - matrix giving the probabilities for the chain to be in each one of the states in the phase \ddot{o}_i , $i = 0,1,2,\dots$.

Then , since d_1 is always the starting state, we have that $P_0 = [1 \ 0 \ 0 \ 0]$

Further it is well known that $P_{i+1} = P_i A$ and therefore

$P_1 = P_0 A = [0 \ 1 \ 0 \ 0]$, $P_2 = P_1 A = [p_{21} \ 0 \ p_{23} \ 0]$, $P_3 = P_2 A = [0 \ p_{21} + p_{23}p_{32} \ 0 \ p_{23}p_{34}]$,

$P_4 = P_3 A = [p_{21}^2 + p_{21}p_{23}p_{32} \ 0 \ p_{21}p_{23} + p_{23}^2 p_{32} \ p_{23}p_{34}]$ (2) and so on .

In general an inductive argument shows that $P_n = P_0 A^n$, $n = 1,2,3,\dots$.

We shall now bring A to its **standard form** A^* by listing the absorbing state first and then partition A^* as follows

$d_4 \ d_1 \ d_2 \ d_3$

$$A^* = \begin{matrix} d_4 \\ d_1 \\ d_2 \\ d_3 \end{matrix} \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & p_{21} & 0 & p_{23} \\ p_{34} & 0 & p_{32} & 0 \end{array} \right].$$

Let $Q = \begin{bmatrix} 0 & 1 & 0 \\ p_{21} & 0 & p_{23} \\ 0 & p_{32} & 0 \end{bmatrix}$ be the matrix of non absorbing states and denote by I_3 the 3x3 unitary matrix, then it is well known that $I_3 - Q$ is always an invertible matrix (e.g. see [3], section 2).

The **fundamental matrix** N of the chain is defined to be

$$N = (I_3 - Q)^{-1} = \frac{1}{D(I_3 - Q)} \text{adj}(I_3 - Q),$$

where $D(I_3 - Q)$ denotes the determinant and $\text{adj}(I_3 - Q)$ denotes the adjoint matrix of $I_3 - Q$.

Thus a straightforward calculation and the use of relation (1) give that

$$N = \frac{1}{p_{23}p_{34}} \begin{bmatrix} 1 - p_{32}p_{23} & 1 & p_{23} \\ p_{21} & 1 & p_{23} \\ p_{21}p_{32} & p_{32} & p_{23} \end{bmatrix} = [n_{ij}], \quad i, j = 1, 2, 3 \quad (3)$$

We recall that the ij -th entry of N gives the mean number of times in state d_j before the absorption, when the chain is started in d_i . Therefore, since d_1 is always the starting state, the mean number of steps taken before absorption is given by

$$t = \sum_{i=1}^3 n_{1i} = \frac{2 + p_{23}p_{34}}{p_{23}p_{34}} \quad (4).$$

It is clear that the bigger the value of t , the more the difficulties that the decision maker faces during the decision making process; in other words we have obtained a measure for the difficulty of the decision making process.

3. Examples

D) The management of a wine - production company A has hired a group of specialists in order to decide about the suitable place for building up a new factory and has agreed to pay them for 6 working hours (w.h.) each time where an analysis of the decision - problem is required , for 54 w.h. each time where collection and interpretation of necessary information is needed , for 28 w.h. whenever a determination of the feasible solutions is needed and for 9 w.h. for the final choice of the best solution .

We shall describe the d-m process in terms of the model presented in the previous section and we shall calculate the expected total sum of w.h. to be paid , the mean number of steps needed before taking the decision and the probability for the d-m process to be terminated in 4 steps .

d_1 : Analysis of the decision problem

It turns out that the profitability of the decision depends upon the types of wine , say

W_1, W_2, \dots, W_n , which are going to be produced by the antagonist companies, say

B_1, B_2, \dots, B_m

d_2 : Collection and interpretation of the necessary information

Suppose that the market's research has shown that there is only an antagonist company, say B_1 , which has the potential to produce three different types of wine , say W_1, W_2 , and W_3 d_3 : Determination of the feasible solutions

Assume that the general situation of the area where the company A acts (communications, traffic, the already existing factories and storehouses of the companies A and B_1 etc), combined with the funds available for the construction of the new factory, suggest that there are four favorable places , say P_1, P_2, P_3 and P_4 for the construction of the new factory .

$d_3 \rightarrow d_2$: Going back from d_3 to d_2

Assume that the market's research shows that the expected net profits of the company A with respect to the possible places for the construction of the new factory and the types of the wine which could be produced by the antagonist company are given in table 1 below :

$P_1 P_2 P_3 P_4$

$$\begin{matrix} W_1 \\ W_2 \\ W_3 \end{matrix} \begin{bmatrix} 3 & 8 & 5 & 4 \\ 4 & 2 & 6 & 5 \\ 2 & 1 & 1 & -1 \end{bmatrix} \text{Table 1}$$

$d_2 \rightarrow d_3$: New transition from d_2 to d_3

From table 1 becomes evident that the feasible solution P_4 is worse than P_3 and therefore P_4 is rejected .

d_4 : Choice of the best solution

Assume that the management of the company has no margin to risk for low profits from the new factory and therefore wants to adopt a conservative criterion for the choice of the best place for building it up.

Such a very common criterion, due to Wald and called the maximin of payoffs (profits), suggests to maximize the minimal, for each case, possible profits. According to this criterion , since the minimal expected profit from the choice of P_1 is 2 monetary units, while the corresponding profit from the choice of P_2 or P_3 is 1 monetary unit, the place P_1 must be chosen for building up the new factory .

From the description of the d-m process above it becomes evident that $p_{21} = 0$, $p_{23} = 1$. Also we have that $p_{32} = p_{34} = 0.5$. In fact , when the chain reaches state d_3 for the first time , the probability to go back to d_2 in the next phase is 1 (since collection and interpretation of new information becomes necessary) , while , the second time that the chain reaches d_3 , the probability to reach d_2 in the next phase is 0. Thus the transition probability p_{32} is equal to the average $\frac{0+1}{2}$ and therefore $p_{34} = 1 - p_{32} = 0.5$. Thus, relation (3) gives that

$$N = \frac{1}{0.5} \begin{bmatrix} 0.5 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0.5 & 1 \end{bmatrix} \text{and therefore } n_{11} = 1 , n_{12} = n_{13} = 2 . \text{ Thus the mean number}$$

of steps before taking the decision is $t = 5$ steps , while the expected total sum of w.h. to be paid is $1.6+2.54+2.28+1.9 = 179$ w.h.. Further , relation (2) gives that $P_3 = [0 \ 0.5 \ 0 \ 0.5]$, i.e $p_4^{(3)} = 0.5$ and therefore the probability for the d-m process to be terminated in 4 steps is 50% . This could happen if

there was no feasible solution worse than one of the others and therefore we didn't reject any of them (as we did for P_4).

II) Consider the previous problem and assume that the market's research has shown that there is a 30% probability that a new antagonist company B_2 will appear in the near future, offering better prices for its wines. It is nevertheless expected that the possible appearance of B_2 will not cause dramatical changes to the general situation of the area where A acts and therefore the feasible solutions for the construction of the new factory of A will remain the same .

Under these conditions the management of the company A ,after the initial analysis of the decision-problem , the collection and interpretation of the necessary information and the determination of the feasible solutions, reach to the conclusion that it is better to postpone its decision about the new factory for one year, in order to see what will happen meanwhile. The "flow-diagram" of the d-m process up to this point is represented in figure 2 below :

$$d_1 \rightarrow d_2 \rightarrow d_3 \text{ Figure 2}$$

Then, if B_2 appears, a new analysis of the decision-problem will be needed , where, apart from the antagonist companies B_1 , B_2 and the types of the wines that they can produce, the prices of the wines must be also be taken into consideration.

Thus, the continuation of the "flow-diagram" of figure 2 is represented in figure 3 below :

$$d_3 \leftrightarrow d_2 \leftrightarrow d_1 \text{ Figure 3}$$

↓

d_4

From figures 2 and 3 becomes evident that the transition probability p_{21} is equal to the average $\frac{0+0.3+0}{3} = 0.1$ and therefore $p_{23} = 0.9$. It becomes also evident that $p_{32} = \frac{1+0}{2}$ and therefore $p_{34} = 0.5$.

Then relation (3) gives that $N = \frac{1}{0.45} \begin{bmatrix} 0.55 & 1 & 0.9 \\ 0.1 & 1 & 0.9 \\ 0.05 & 1 & 0.9 \end{bmatrix}$ and therefore $n_{11} = 1.222$, $n_{12} = 2.222$ and $n_{13} = 2$, i.e $t = 5.444$ steps.

Also the total sum of the expected w.h. to be paid is approximately equal to $(1.222).6+(2.222).54+2.28+1.9 = 190.32$ w.h. (all calculations were made with accuracy up to the third decimal point) .

Finally, from relations (2) we get that $P_3 = [0 \ 0.55 \ 0 \ 0.45]$, therefore the probability for the d-m process to be terminated in 4 steps is 45%

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