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Stability and optimality of solutions to differential inclusions via averaging method

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Abstract The averaging method is one of the most used tools to study dynamical systems. With the development of the theory of differential inclusions, a respective generalization of the averaging method followed the steps outlined in the theory of differential equations. Presently, it has been successfully applied to a wide range of problems involving differential inclusions, simplifying the study of the systems under consideration. In this work, the main development trends and methods in the application of the averaging method to the study of stability and optimality of solutions to differential inclusions are surveyed. A detailed list of references is given and some examples of applications are presented.

Keywords Differential inclusions, Averaging method

Mathematics Subject Classification (2000) Primary 34A60; Secondary 34C29

1 Introduction

The averaging method is one of the most powerful tools used to analyze differential equations appearing in the study of non-linear problems. The idea behind the averaging method consists in replacing the original equation

$$\dot{x} = \epsilon f(t, x), \tag{1}$$

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where ϵ is a small parameter, by an averaged one,

$$\dot{\overline{x}} = \epsilon \overline{f}(\overline{x}) = \epsilon \lim_{T\to\infty} \frac{1}{T} \int_0^T f(s, \overline{x}) ds. \tag{2}$$

This new equation is autonomous, allowing an easier study and characterization. These techniques were applied to solve a wide range of problems.

For differential equations, the first rigorous justification of the method was given by Fatou [15] for some special cases of periodic vector fields. His result was later generalized by Bogoliubov, see, e.g., Bogoliubov and Mitropolsky [3], and is currently known as the first Bogoliubov theorem. Following fundamental Bogoliubov's ideas, the averaging method was developed and extended by many authors, forming today a solid theory with applications in many branches of nonlinear analysis, see, e.g., Sanders et al. [108] and Burd [4].

The first Bogoliubov theorem guarantees that solutions of (1) and (2) remain close at least for $t \in [0, 1/\epsilon]$. Other questions arise naturally when applying the averaging method. Namely, in many situations it is important to know the conditions when it is possible to extend the previous result to an infinite time interval and what correspondence can be establish between proprieties of solutions to equations (1) and (2). Under certain conditions, it is possible to obtain some answers to the previous questions (see, e.g., the books by Sanders et al. [108] and Burd [4], and the paper by Samoilenko & Stanzhitskii [107]).

As the theory of differential inclusions grew, it encountered many applications in different areas, ranging from control theory to discontinuous dynamical systems. With them, came the need to adapt some of the methods developed in the theory of differential equations to differential inclusions. The first generalization of the first Bogoliubov theorem to differential inclusions appeared at the end of 70s of the last century, in two papers by Victor Plotnikov [91,92]. In fact, averaging method for differential inclusions suffered a major development with the works of Plotnikov and his school. For an account of this results, we refer the reader to the books Plotnikov [99] and Plotnikov et al. [94], and a recent survey of Plotnikov's research, Klymchuk et al. [61].

Since Plotnikov's first results, many authors contributed to the development of the averaging method for differential inclusions. In this survey we give a detailed account of the main development trends and methods in the application of the averaging method to the study of stability and optimality of solutions to differential inclusions. We emphasize the results *not* explored in the survey by Klymchuk et al. [61]. More precisely, the paper is organized as follows. In the second section we consider several extensions of the fundamental result, Bogoliubov's first theorem, to differential inclusions under various sets of hypotheses. In the third section we give an account of the various ramifications of the averaging method for differential inclusions. Section 4 is devoted to the presentation of results extending the averaging method to the infinite time interval and the relation between qualitative proprieties of both averaged and original inclusions. The fifth, and last section, concerns the application of the method in the study of optimization problems for differential inclusions.

Throughout this paper we denote by $\mathbb{R}^n$ the real n-dimensional space. The set of non-negative real numbers is denoted by $\mathbb{R}_+$. We use the notations $\langle\cdot,\cdot\rangle$ and $|\cdot|$ for the usual inner product and Euclidean norm, respectively. The set of all compact subsets and all convex compact sets from $\mathbb{R}^n$ are denoted by $\mathcal{K}(\mathbb{R}^n)$ and $\mathcal{KK}(\mathbb{R}^n)$, respectively. We denote by $\mathcal{F}(\mathbb{R}^n)$ the set of all closed subsets of $\mathbb{R}^n$. We use the notation $B=\{x\in\mathbb{R}^n \mid |x|\le 1\}$ for the closed unit ball in $\mathbb{R}^n$. The convex hull and the closure of a subset $S\subset\mathbb{R}^n$ are denoted by $\mathrm{co}S$ and $\mathrm{cl}S$, respectively. Let $\lambda\in\mathbb{R}$. Then put by definition $\lambda S=\{\lambda x \mid x\in S\}$. For two sets A_1 and A_2 in $\mathbb{R}^n$ their sum is defined by $A_1+A_2=\{a_1+a_2 \mid a_1\in A_1,\ a_2\in A_2\}$. The Hausdorff distance between two compact sets $A_1,\,A_2\subset\mathbb{R}^n$ is defined as

$$d_H(A_1,A_2)=\min\{h\ge 0 \mid A_1\subset A_2+hB, A_2\subset A_1+hB\}.$$

The support function of a set $A\subset\mathbb{R}^n$ is denoted by $S(A,\psi)=\sup\{\langle a,\psi\rangle \mid a\in A\}$. We denote by $\mathcal{S}_{[0,L]}(F,x_0)$ the set of solutions to the Cauchy problem $\dot{x}\in F(t,x)$, $t\in[0,L]$, $x(0)=x_0$, and by $\mathcal{R}(t)(F,x_0)=\{x(t) \mid x(\cdot)\in \mathcal{S}_{[0,t]}(F,x_0)\}$ the reachability set. We use also the notations $\mathcal{S}_{[0,L]}(F,C)=\bigcup_{x_0\in C}\mathcal{S}_{[0,L]}(F,x_0)$ and $\mathcal{S}_{[0,L]}(F)=\mathcal{S}_{[0,L]}(F,\mathbb{R}^n)$. The closed unit ball in the space of continuous functions $f:[0,L]\to\mathbb{R}^n$ with the uniform norm, $C([0,L],\mathbb{R}^n)$, is denoted by $\mathcal{B}$. The set of locally integrable functions $f:[0,\infty[\to\mathbb{R}^n$ is denoted by $L_1^{\mathrm{loc}}([0,\infty[,\mathbb{R}^n)$. The transpose of a matrix C is denoted by C^*. The set of positively definite symmetric $n\times n$-matrices is denoted by $M(n)$.

2 Averaging on a finite time interval

In this section we present various versions of Bogoliubov's first theorem for differential inclusions. Different conditions imposed on the right-hand side (r.h.s.) are discussed. Consider a differential inclusion

$$\dot{x}\in\epsilon F(t,x), \tag{3}$$

where $\epsilon>0$ is a small parameter and $F:\mathbb{R}\times\mathbb{R}^n\to\mathcal{K}(\mathbb{R})$ is a set-valued map. Assume that there exists the limit

$$\overline{F}(x)=\lim_{T\to\infty}\frac{1}{T}\int_0^T F(s,x)ds, \tag{4}$$

i.e.,

$$\lim_{T\to\infty} d_H\left(\overline{F}(x),\frac{1}{T}\int_0^T F(s,x)ds\right)=0.$$

(Here and in what follows the set-valued integral is the Lebesgue/Aumann integral, see, e.g., Castaing & Valadier [5].) Then we define the averaged inclusion

$$\dot{\overline{x}}\in\epsilon\overline{F}(\overline{x}), \tag{5}$$

associated with inclusion (3).

In this section we consider set-valued maps satisfying various properties in domains $D \subset \mathbb{R}^n$ and $D' \subset \mathbb{R}^n$. It is assumed that the inclusion $D' + \delta B \subset D$ holds for some $\delta > 0$.

2.1 Differential inclusions with a Lipschitzian r.h.s.

In the case of a Lipschitzian in x r.h.s. $F(t,x)$, the set-valued version of Bogoliubov's first theorem was proved by Plotnikov in [92].

Theorem 1 ([92]) *Suppose that the following conditions hold in the domain* $Q = \{t \geqslant 0,\ x \in D \subset \mathbb{R}^n\}$*:*

1. *the set-valued map $F(t,x)$ with compact values is continuous, uniformly bounded, and satisfies the Lipchitz condition with respect to x with constant λ, i.e.,*

$$F(t,x) \subset MB_n, \qquad d_H(F(t,x_2), F(t,x_1)) \leqslant \lambda |x_2 - x_1|;$$

2. *the limit (4) exists in the domain D uniformly with respect to x;*
3. *for all $x_0 \in D'$ and $\overline{x}(\cdot) \in \mathcal{S}_{[0,+\infty)}(\epsilon F, x_0)$, there exists $\rho > 0$ such that the inclusion $x(t) + \rho B \subset D$, $t \geq 0$, holds.*

Then, for any $\eta > 0$ and $L > 0$ there exists $\epsilon_0(\eta, L) > 0$ such that the following statements are valid for $\epsilon \in (0, \epsilon_0]$ on the interval $0 \leqslant t \leqslant L/\epsilon$:

(a) for any solution $\overline{x}(\cdot) \in \mathcal{S}_{[0,L/\epsilon]}(\epsilon \overline{F}, x_0)$ there exists a solution $x(\cdot) \in \mathcal{S}_{[0,L/\epsilon]}(\epsilon F, x_0)$ satisfying

$$|x(t) - \overline{x}(t)| < \eta, \quad t \in [0, L/\epsilon]; \tag{6}$$

(b) for any solution $x(\cdot) \in \mathcal{S}_{[0,L/\epsilon]}(\epsilon F, x_0)$ there exists a solution $\overline{x}(\cdot) \in \mathcal{S}_{[0,L/\epsilon]}(\epsilon \overline{F}, x_0)$ such that inequality (6) holds.

Thus we have the estimate

$$d_H(\mathrm{cl}\mathcal{R}(t)(\epsilon F, x_0), \mathcal{R}(t)(\epsilon \overline{F}, x_0)) < \eta, \tag{7}$$

for $0 < \epsilon < \epsilon_0$ and $t \in [0, L/\epsilon]$.

If, besides satisfying the conditions of the last theorem, the set-valued map $F(t,x)$ is periodic in t, it is possible to prove (see Plotnikov [91]) that for any $L > 0$, there exist $\epsilon_0(L) > 0$ and $C(L) > 0$ such that

$$d_H(\mathrm{cl}\mathcal{R}(t)(\epsilon F, x_0), \mathcal{R}(t)(\epsilon \overline{F}, x_0)) < C(L)\epsilon, \tag{8}$$

for $0 < \epsilon < \epsilon_0$ and $t \in [0, L/\epsilon]$, obtaining a more precise estimate.

Theorem 1 was generalized afterwards by Vasil'ev [122], to set-valued maps $F(t,x)$ measurable in t and satisfying a more general Lipschtz condition:

$$d_H(F(t,x_2), F(t,x_1)) \leqslant \lambda(t)|x_2 - x_1|, \quad \text{for all} \quad x_1, x_2 \in D,$$

where $\lambda(\cdot)$ is an integrable function such that

$$\sup\left\{(t_2-t_1)^{-1}\int_{t_1}^{t_2}\lambda(s)ds\;:\;t_1<t_2\right\}<+\infty.$$

These cornerstone results were extended in several directions. In the paper [60], Klimchuk presented an extension of Plotnikov's results from [91] to unbounded set-valued maps, $F:\mathbb{R}\times\mathbb{R}^n\to\mathcal{F}(\mathbb{R})$, assuming convexity of the set $F(t,x)$ for all $(t,x)\in Q$ and the inclusion $0\in F(t,x)$. In Plotnikov [93] it was considered a partial averaging scheme, extending Theorem 1 to this case.

In general, the limit (4) does not exist. To overcome this difficulty, Plotnikov & Savchenko [100] suggested an extension of Theorem 1. Under the following conditions:

(I) the set-valued map $F(t,x)$ is measurable in t and there exist integrable functions $M(\cdot)$ and $\lambda(\cdot)$ and constants M and λ such that

$$F(t,x)\subset M(t)B_n,\qquad d_H(F(t,x_2),F(t,x_1))\leqslant\lambda(t)|x_2-x_1|,$$

and, for any finite segment $[t_1,t_2]\subset[0,+\infty[$, the inequalities

$$\int_{t_1}^{t_2}M(s)ds\leqslant M(t_2-t_1),\qquad\int_{t_1}^{t_2}\lambda(s)ds\leqslant\lambda(t_2-t_1),$$

hold;

(II) there exist Lipschitzian and bounded set-valued maps $\overline{F}_+,\overline{F}_-:D\to\mathcal{KK}(\mathbb{R}^n)$ satisfying, for all $\eta>0$, the inclusions

$$\begin{aligned}&\overline{F}_-(x)\subset\frac{1}{T}\int_0^T F(s,x)ds+\eta B_n,\\&\frac{1}{T}\int_0^T F(s,x)ds\subset\overline{F}_+(x)+\eta B_n,\end{aligned}\tag{9}$$

whenever $T>0$ is sufficiently large. Moreover, condition (9) is satisfied uniformly with respect to $x\in D$;

(III) for all $x_0\in D'$, solutions of inclusions

$$\begin{aligned}&\dot{\overline{x}}_-\in\epsilon\overline{F}_-(\overline{x}_-),\quad\overline{x}_-(0)=x_0,\\&\dot{\overline{x}}_+\in\epsilon\overline{F}_+(\overline{x}_+),\quad\overline{x}_+(0)=x_0,\end{aligned}$$

together with a certain ρ-neighborhood lie in D;

they proved that for all $\eta>0$ and $L>0$ there exists $\epsilon_0(\eta,L)$ such that the inclusions

$$\mathcal{R}(t)(\epsilon\overline{F}_-,x_0)\subset\mathcal{R}(t)(\epsilon F,x_0)+\eta B_n,\;\;t\in[0,L/\epsilon]$$

and

$$\mathcal{R}(t)(\epsilon F,x_0)\subset\mathcal{R}(t)(\epsilon\overline{F}_+,x_0)+\eta B_n,\;\;t\in[0,L/\epsilon]$$

hold for all $\epsilon\in(0,\epsilon_0(\eta,L)]$.

2.2 Differential inclusions with a generalized Lipschitzian r.h.s.

The proofs of the previous results, essentially use the Filippov theorem from [26] to obtain the closeness estimates, imposing the need to consider inclusion (3) with Lipschitzian in x r.h.s.. In some studies, this condition was weakened allowing to embrace broader classes of differential inclusions.

Following the paper by Donchev & Farkhi [11], where the authors presented a generalization of the Filippov theorem for differential inclusions with one-sided Lipschitzian r.h.s., several results were obtained extending the averaging method to this case. Recall the definition of one-sided Lipschitzian set-valued map.

Definition 1 A set-valued map $F : \mathbb{R} \times \mathbb{R}^n \to \mathcal{K}(\mathbb{R})$ is called one-sided Lipschitz continuous (with respect to x) if there is a locally integrable function $\lambda \mathbb{R} \to \mathbb{R}$ such that for every $x, y \in \mathbb{R}^n,\ t \in \mathbb{R}$, and $v \in F(t,x)$, there exists $w \in F(t,y)$ satisfying

$$\langle x - y, v - w \rangle \leqslant \lambda(t)|x - y|^2.$$

This class of set-valued maps is less restrictive than the class of Lipschitzian ones, containing even some discontinuous maps. In this case, a rigorous justification of the averaging method was given by Sokolovskaya [117]. Under the same one-sided Lipschitz condition, other extensions of the method were obtained in the papers by Donchev et al. [13], for perturbed one sided Lipschitz differential inclusions, by Donchev & Grammel [12], for functional differential inclusions in Banach spaces, and by Donchev [10] for evolution inclusions in Banach spaces. In the papers by Sokolovskaya [114–116] and Sokolovskaya & Filatov [119] some approximation theorems were proved using the averaging method for inclusions with r.h.s. satisfying one-sided Lipschitzian condition.

2.3 Differential inclusions with a continuous r.h.s.

Some results were also obtained with no use of Lipschitz or generalized Lipschitz conditions assuming only continuity of the r.h.s. with respect to x. For differential inclusions (even differential equations) with continuous r.h.s. estimate (7) cannot be obtained even in very simple cases, as we can see from the following example.

Example 1 Consider the equation

$$\dot{x} = \epsilon(\sqrt{|x|} + \sin t). \tag{10}$$

The respective averaged equation obviously is

$$\dot{\overline{x}} = \epsilon\sqrt{|\overline{x}|}.$$

Observe that

$$x(t + \tau) \geq x(t) + \epsilon \int_t^{t+\tau} \sin s ds = x(t) - \epsilon(\cos(t + \tau) - \cos t). \tag{11}$$

Consider a solution to (10) satisfying $x(0) = 0$ and set $x_n = x(2\pi n)$. Show that

$$x_n \geq \frac{\pi^2 n^2 \epsilon^2}{32}, \tag{12}$$

whenever $\epsilon < 16^2/\pi^2$. Indeed, from (11) we have

$$x(2\pi n + \pi/2) \geq x_n + \epsilon. \tag{13}$$

Since $\sin t \geq 0$, $t \in [2\pi n+\pi/2, 2\pi n+\pi]$, we get $\dot{x}(t) \geq 0$, $t \in [2\pi n+\pi/2, 2\pi n+\pi]$, and therefore $x(t) \geq x(2\pi n + \pi/2)$, $t \in [2\pi n + \pi/2, 2\pi n + \pi]$. From this and (13) we obtain $\dot{x}(t) \geq \epsilon(\sqrt{x_n + \epsilon} + \sin t)$. Integrating this inequality, from (13) we have

$$x(2\pi n + \pi) \geq x(2\pi n + \pi/2) + \frac{\pi}{2}\epsilon\sqrt{x_n + \epsilon} + \epsilon \geq x_n + \frac{\pi}{2}\epsilon\sqrt{x_n + \epsilon} + 2\epsilon.$$

Applying (11) again we obtain

$$x_{n+1} = x(2\pi n + 2\pi) \geq x(2\pi n + \pi) - 2\epsilon \geq x_n + \frac{\pi}{2}\epsilon\sqrt{x_n + \epsilon}.$$

Now, if $\epsilon < 16^2/\pi^2$, we get inequality (12) by induction from the following inequalities:

$$\frac{\pi^2\epsilon^2}{4}\left(\frac{\pi^2 n^2\epsilon^2}{32} + \epsilon\right) \geq \frac{8\pi^4 n^2\epsilon^4}{32^2} + \frac{\pi^4\epsilon^4}{32^2},$$

$$\frac{8\pi^4 n^2\epsilon^4}{32^2} + \frac{\pi^4\epsilon^4}{32^2} \geq \frac{\pi^4\epsilon^4}{32^2}(2n+1)^2,$$

$$\frac{\pi^2 n^2\epsilon^2}{32} + \frac{\pi\epsilon}{2}\sqrt{\frac{\pi^2 n^2\epsilon^2}{32} + \epsilon} \geq \frac{\pi^2 n^2\epsilon^2}{32} + \frac{\pi^2\epsilon^2}{32}(2n+1) = \frac{\pi^2(n+1)^2\epsilon^2}{32}.$$

Set $\epsilon = 1/(2\pi n)$. Then we have $x(2\pi n) \geq 1/128$, i.e., the trivial solution $\overline{x}_0(t) \equiv 0$ to the averaged equation cannot be approximated by solutions of (10) in order to satisfy the condition $|x(1/\epsilon) - \overline{x}_0(1/\epsilon)| < \epsilon$.

In [103], Plotnikova proved a set-valued version of the Krasnosel'skii-Krein theorem, [68]. By analogy with ordinary differential equations, she explored the relation between the averaging method on a finite interval and the continuous dependence of the solutions to differential inclusions with respect to a parameter, obtaining a unilateral version of Bogolyubov's first theorem for inclusions with continuous right-hand side.

Theorem 2 ([103]) *Let $D \subset \mathbb{R}^n$ be a bounded domain. Suppose that the following conditions hold in the domain $Q = \{t \geqslant 0,\ x \in D\}$:*

1. *the map $F(t, x)$ with compact convex values is uniformly bounded, continuous in t and uniformly continuous in x with respect to t;*
2. *the limit (4) exists in the domain D;*

3. *for all* $x_0 \in D'$ *and* $\tau \in [0, L]$ *the solutions of the problem*

$$\frac{dy}{d\tau} \in \overline{F}(y), \quad y(0) = x_0 \tag{14}$$

are defined and, together with a ρ*-neighborhood, lie in the domain* D.

Then, for any $\eta > 0$ *and* $L > 0$ *there exists* $\epsilon_0 = \epsilon_0(\eta, L) > 0$ *such that for all* $\epsilon \in (0, \epsilon_0]$ *and for each solution* $x(\cdot) \in \mathcal{S}(\epsilon F, x_0)$ *there is a solution* $y(\cdot) \in \mathcal{S}_{[0,L]}(\overline{F}, x_0)$ *satisfying* $|x(t) - y(\epsilon t)| < \eta$, $t \in [0, L/\epsilon]$.

Plotnikova also proved a similar result for so-called R-solutions [76] of differential inclusions [103].

A very interesting approach to this problem can be found in Lakrib [69] where the justification of the averaging method for differential inclusions with continuous r.h.s. on finite time intervals was given using nonstandard analysis.

2.4 Differential inclusions with a discontinuous r.h.s.

For many problems, the classes of set-valued maps used in the cited papers are too restrictive. Indeed, in various practical applications the dynamical systems are discontinuous, not allowing, in general, the justification of the application of the averaging method by the previous results. Different problems described by differential equations with discontinuous r.h.s. were studied in [70, 78, 59, 101, 102, 16].

In general, for those systems, even the classical notion of solution cannot be used and we have to resort to the theory of differential inclusions. At the end of 50s, Filippov proposed a generalized concept of solution, rewriting the problem under consideration as a differential inclusion, where the set-valued map obtained via Filippov regularization, is upper semi-continuous in x (see, e.g., Filippov [27]). The Filippov definition of solution guarantees the existence of a solution to the Cauchy problem for any initial condition. For differential equations with a continuous right-hand side it coincides with the usual one, and the properties of the solution set, like connectedness and compactness, are the same. This definition allows one to develop a rigorous theory of discontinuous systems, including averaging method.

In Plotnikov [95], a version of the first Bogoliubov theorem was obtained for differential inclusions with periodic upper semi-continuous right-hand side.

Theorem 3 ([95]) *Suppose that the following conditions hold in the domain* $Q = \{t \geqslant 0,\ x \in D \subset \mathbb{R}^n\}$:

1. *the map* $F(t, x)$ *with convex compact values is upper semi-continuous with respect to* x, 2π*-periodic and measurable with respect to* t, *and uniformly bounded by some integrable function* $M(t)$, *satisfying* $\int_{t_1}^{t_2} M(s)ds \leqslant M(t_2 - t_1)$, *for all* $t_2 > t_1 > 0$;
2. *the map* $\overline{F}(x)$ *satisfies Lipschitz condition with constant* λ;

3. for all $x_0 \in D'$ *and* $\overline{x}(\cdot) \in \mathcal{S}_{[0,+\infty)}(\epsilon F, x_0)$*, there exists* $\rho > 0$ *such that the inclusion* $x(t) + \rho B \subset D$*,* $t \geq 0$*, holds.*

Then, for any $\eta > 0$ *and* $L > 0$ *there exists* $\epsilon_0(\eta, L) > 0$ *such that the following inclusion holds:*

$$\mathcal{S}_{[0,L/\epsilon]}(\epsilon F, x_0) \subset \mathcal{S}_{[0,L/\epsilon]}(\epsilon \overline{F}, x_0) + \eta \mathcal{B}, \quad 0 < \epsilon < \epsilon_0(\eta, L). \tag{15}$$

Although considering upper semi-continuous r.h.s., the need for a Lipschitzian averaged inclusion, reduces the applicability of the result.

In the same paper [95] Plotnikov also considered some special cases of upper semi-continuous differential inclusions, for which it is possible to establish estimates of the form (7). For example, he considered piecewise-continuous Lipschitzian set-valued maps of the form

$$F(t,x) = \begin{cases} F_1(t,x), \, x \in D_1 \\ F_2(t,x), \, x \in D_2 \end{cases}, \tag{16}$$

where

$$D_1 = \{x \in \mathbb{R}^n \mid \phi(x) > 0\} \text{ and } D_2 = \{x \in \mathbb{R}^n \mid \phi(x) < 0\}$$

and justified the application of the averaging method in the absence of sliding of the trajectories over the surface of discontinuity.

Some other results for upper semi-continuous inclusions were also obtained by Klimov [64] and by Klimov & Ukhalov [65], imposing certain one-sided constraints on the r.h.s of inclusion (3).

The most general result for inclusions with upper semi-continuous r.h.s. appeared in Gama, Guerman & Smirnov [34]. In this paper the authors introduced new averaged inclusion allowing to generalize the averaging method to differential inclusions with upper semi-continuous r.h.s. without additional assumptions. They defined the averaged differential inclusion as

$$\dot{x} \in \overline{F}(x) = \bigcap_{\delta > 0} \overline{F}^{\delta}(x), \tag{17}$$

where $\overline{F}^{\delta}(x)$ is the convex hull of the map

$$\overline{\Phi}^{\delta}(x) = \limsup_{\theta \uparrow 1} \limsup_{T \to \infty} \frac{1}{(1-\theta)T} I(\theta T, T, x, \delta), \tag{18}$$

and

$$I(t_1, t_2, x, \delta) = \left\{ \int_{t_1}^{t_2} v(t) dt \mid v(\cdot) \in L_1^{\text{loc}}([0, \infty[, \mathbb{R}^n), \; v(t) \in F(t, x + \delta B) \right\}.$$

The $\limsup$ stands for the Kuratowski upper limit, i.e., the set of all limit points.

If the set-valued map $t \to F(t,x)$ is 2π-periodic and bounded, then

$$\overline{F}(x) = \frac{1}{2\pi}\int_0^{2\pi} F(t,x)dt.$$

Under Lipschitz condition this map coincides with (4), if the limit exists. Note that the map $\overline{F}$ cannot be substituted by the map $\overline{\Phi}^0$, if we have to guarantee the existences of solutions to the averaged differential inclusion. To illustrate this consider the following example:

Example 2 ([34]) Let $F : [0,\infty[\times\mathbb{R} \to \mathbb{R}$ be given by

$$F(t,x) = \begin{cases} \{1\}, & x \in]-\infty, -2/(t+1)[, \\ \{-1-(t+1)x\}, & x \in [-2/(t+1), 1/(t+1)], \\ \{-2\}, & x \in]1/(t+1), \infty[. \end{cases}$$

Obviously we have

$$\overline{\Phi}^0(x) = \begin{cases} \{1\}, & x < 0, \\ \{-1\}, & x = 0, \\ \{-2\}, & x > 0. \end{cases}$$

The Cauchy problem $\dot{x} \in \overline{\Phi}^0(x)$, $x(0) = 0$, has no solution. On the other hand, the map

$$\overline{F}(x) = \begin{cases} \{1\}, & x < 0, \\ [-2,1], & x = 0, \\ \{-2\}, & x > 0, \end{cases}$$

is convex-valued and the Cauchy problem $\dot{x} \in \overline{F}(x)$, $x(0) = 0$, has a solution.

Using this averaging operator and assuming that the set-valued map $F : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ satisfies the following conditions:

(C1) $F(t,x) \subset \mathcal{KK}(\mathbb{R}^n)$, for all $(t,x) \in \mathbb{R}\times\mathbb{R}^n$;
(C2) the set-valued map $F(t,\cdot)$ is upper semi-continuous;
(C3) for any x there exists measurable selection of $F(t,x)$, that is, there exists $f(t,x) \in F(t,x)$ such that $t \to f(t,x)$ is measurable for all x;
(C4) there exists a nonnegative $b(\cdot) \in L_1^{\mathrm{loc}}([0,\infty[,\mathbb{R})$ such that $F(t,x) \subset b(t)B$ for all $(t,x) \in [0,+\infty[\times\mathbb{R}^n$;
(C5) there exists the limit

$$b = \lim_{T\to\infty} \frac{1}{T}\int_0^T b(t)dt;$$

the authors proved the following version of Bogolyubov's first theorem:

Theorem 4 ([34]) *Let $L > 0$ and let $F : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ be a set-valued map satisfying conditions (C1) - (C5). Let $C \in \mathcal{K}(\mathbb{R}^n)$. Then for any $\eta > 0$ and $L > 0$ there exists $\epsilon_0 = \epsilon_0(\eta, L) > 0$ such that for any $\epsilon \in]0,\epsilon_0[$ and any solution $x(\cdot) \in \mathcal{S}_{[0,L/\epsilon]}(\epsilon F, C)$, there exists a solution $\overline{x}(\cdot) \in \mathcal{S}_{[0,L/\epsilon]}(\epsilon\overline{F}, C)$ satisfying*

$$|x(t) - \overline{x}(t)| < \eta, \quad t \in [0, L/\epsilon].$$

The main tool used in the proof of this result was an approximation theorem presented in the paper. The approximation theorem guarantees that, under conditions (C1) - (C4), the set of solutions to a non-autonomous differential inclusion with upper semi-continuous in x r.h.s. can be represented as an intersection of a decreasing sequence of sets of solutions to Lipschitzian differential inclusions.

Averaged inclusion (17) allows unilateral estimate even when limit (4) does not exist.

Example 3 ([34]) Consider the differential inclusion

$$\dot{x}(t) \in \epsilon F(t) = \epsilon\{\sin\log(t+1)\}.$$

Obviously the limit

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T F(t)dt$$

does not exist. However, $\bar{F} \equiv [-1/\sqrt{2}, 1/\sqrt{2}]$, and we have the inclusion

$$\mathcal{S}_{[0,L/\epsilon]}(\epsilon F, x_0) \subset \mathcal{S}_{[0,L/\epsilon]}(\epsilon\overline{F}, x_0) + \eta\mathcal{B}, \;\; 0 < \epsilon < \epsilon_0(\eta, L).$$

Estimate (7), clearly, does not hold.

3 Other averaging problems on a finite time interval

The theory of differential inclusions is a mature and still growing field. Following its development, in the last two decades, several papers appeared adapting the averaging techniques to various branches of the theory of differential inclusions. In this section we present a detailed description of the various generalizations of the method.

3.1 Systems of inclusions with fast and slow variables

In Khapaev & Filatov [23,24] the averaged method was extended to systems of differential inclusions with fast and slow variables

$$\begin{aligned} &\dot{x} \in \epsilon F(t, x, y, \epsilon), \quad x(0) = x_0, \\ &\dot{y} \in G(t, x, y, \epsilon), \quad y(0) = y_0, \end{aligned} \tag{19}$$

where, $\epsilon > 0$ is a small parameter and F and G are Lipschitzian in (x, y). The averaged set-valued map $\overline{F} : \mathbb{R}^n \to \mathcal{KK}(\mathbb{R}^n)$ is a Lipschitzian map satisfying the following condition: for all $\eta > 0$ there exists $T(\eta) > 0$, such that for all x_0, y_0, $t_0 \geqslant 0$, $T \geqslant T(\eta)$, and $\overline{y}(\cdot) \in \mathcal{S}_{[0,\infty[}(\overline{G}_{x_0}, (y_0))$, where $G_{x_0}(t, y) \equiv G(t, x_0, y, 0)$, the inclusion

$$\frac{1}{T}\int_{t_0}^{t_0+T} F(s, x_0, \overline{y}(s), 0)ds \subset \overline{F}(x_0) + \eta B \tag{20}$$

holds. Khapaev & Filatov showed that for any $\eta > 0$ there exists $\epsilon_0 > 0$ such that if $\epsilon \in]0, \epsilon_0[$, then for any solution $(x(\cdot), y(\cdot))$ to system (19) there exists a solution $\overline{x}(\cdot) \in \mathcal{S}_{[0,L/\epsilon]}(\epsilon\overline{F}_0, x_0)$ satisfying

$$|x(t) - \overline{x}(t)| < \eta, \quad t \in [0, L/\epsilon].$$

Some generalizations to one-sided Lipschitzian inclusions are presented in Sokolovskaya [118]. Filatov, in [18], obtained some estimates for support functions of averaged maps, which facilitate the application of the method. Kairakbaev used these techniques to the study of gyroscope motion (see [50]). Control systems with structure (19) were considered by Filatov in [19]. The Khapaev & Filatov results are summarized in their monograph [25]. Systems (19) with set-valued maps satisfying one-sided Lipschitz condition with respect to x and y were considered by Filatov in [21, 22].

3.2 Impulsive Differential Inclusions

For differential inclusion with jump conditions at discrete moments of time

$$\begin{aligned} &\dot{x} \in \epsilon F(t, x), \quad t \neq \tau_i, \quad x(0) = x_0, \\ &\Delta x\big|_{t=\tau_i} \in \epsilon I_i(x) \end{aligned} \tag{21}$$

where $\epsilon > 0$, τ_i are instants of the pulse action, and $I_i : \mathbb{R}^n \to \mathcal{KK}(\mathbb{R})$ are set-valued maps, the averaged r.h.s. is defined by

$$\overline{F}(x) = \lim_{T\to\infty} \left(\frac{1}{T} \int_t^{t+T} F(s, x) ds + \frac{1}{T} \sum_{t \leqslant \tau_i < t+T} I_i(x) \right). \tag{22}$$

The convergence in (22) is understood in the sense of the Housdorff metric. The justification of the averaging method for such systems in terms of R-solutions was given by Plotnikov & Plotnikova [98]. A detailed account of the works on this subject can be found in the book by Perestyuk et al. [77] and in Klymchuk et al. [61]. Skripnik [109] extended the averaging method to impulsive differential inclusions with Hukuhara derivative.

3.3 Singularly Perturbed Systems

Consider a singularly perturbed differential inclusion

$$\dot{z}(t) \in F(z(t), y(t)), \qquad \epsilon\dot{y}(t) \in G(z(t), y(t)), \tag{23}$$

where $z(\cdot)$ and $y(\cdot)$ are the slow and fast motions, respectively, and $\epsilon > 0$ is a small singular perturbation parameter. After the time transformation $\tau = t/\epsilon$, inclusions (23) can be written in the form

$$\dot{z}(\tau) \in \epsilon F(z(\tau), y(\tau)), \qquad \dot{y}(\tau) \in G(z(\tau), y(\tau)), \tag{24}$$

with unchanged initial conditions,

$$z(0) = z_0 \in \mathbb{R}^m, \qquad y(0) = y_0 \in \mathbb{R}^n.$$

Consider the Cauchy problem

$$\dot{y}(\tau) \in G(z, y(\tau)), \qquad y(0) = y_0 \in \mathbb{R}^n, \tag{25}$$

where the slow state $z \in \mathbb{R}^m$ is fixed. Let $y_z(\cdot, y_0)$ be a solution of (25). The large time behavior of the trajectories of system (24) can be studied applying the averaging method. Grammel [39] defined the finite time average by

$$F_S(z, y_0) = \mathrm{cl} \bigcup_{y_z(\cdot, y_0)} \frac{1}{S} \int_0^S F(z, y_z(\tau, y_0)) d\tau.$$

Assuming, in addition to some other technical conditions, that there exist a continuous set-valued map $F_0 : \mathbb{R}^m \to \mathbb{R}^m$ with closed, uniformly bounded, convex, non-empty images and a continuous function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\lim_{S \to \infty} \gamma(S) = 0$, such that, for all $z \in \mathbb{R}^m, y_0$, and any $S > 0$ the finite time averages satisfy the following estimate:

$$d_H(F_S(z, y_0), F_0(z)) \leqslant \gamma(S),$$

Grammel proved that the behavior of slow variables of system (24) can be approximately described by the behavior of solutions to the differential inclusion

$$\dot{z}(t) \in F_0(z(t)).$$

An estimate for the order of approximation was obtaind in Grammel [41].

This approach was largely used to study other singularly perturbed systems. In Donchev & Slavov [14,9], singularly perturbed differential inclusions with generalized solutions where considered. Several singularly perturbed control problems where studied in the papers by Gaitsgory [28], Grammel [40], Gaitsgory & Leizarowitz [31], Gaitsgory & Grammel [30], Gaitsgory & Nguyen [32], Gaitsgory & Nguyen [33], Gaitsgory [29], Grammel [43], Wang et al. [124, 125]. Kamenski [54,51,52] applied the averaging method to singularly perturbed systems of semilinear differential inclusions (see also the book by Kamenski et al. [53]).

3.4 Miscellaneous averaging problems

The averaging method has been also applied to a wide variety of problems involving differential inclusions or closely related systems. Here we list the most important topics.

Fuzzy differential inclusions: Bogoliubov's first theorem (Skripnik [111]); averaging of fuzzy impulsive differential inclusions (Skripnik [112]); partial averaging (A. Plotnikov et al. [89]); fuzzy differential inclusions in the case when the average of the r.h.s. does not exist (A. Plotnikov et al. [90]); some related results (A. Plotnikov [85,83], Komleva et al. [67], A. Plotnikov & Komleva [87]).

Parabolic inclusions: Bogoliubov's first theorem for parabolic inclusions (Klimov [63]); periodic solutions to parabolic inclusions with memory (Klimov [62]).

Neutral differential inclusions: neutral differential inclusions in the case when the average of the r.h.s. does not exist (Janiak & Luczak-Kumorek [48]).

Set differential inclusions: averaging of differential inclusions with Hukuhara derivative (Kisielewicz [57], Janiak & Luczak-Kumorek [46], A. Plotnikov [80], Kichmarenko [56], Skripnik [110]); averaging for set differential inclusions in a semilinear metric space when the average of the r.h.d is absent (A. Plotnikov et al. [86]); partial averaging for differential inclusions with Hukuhara's derivative (Janiak & Luczak-Kumorek [49]); averaging of controlled equations with Hukuhara derivative (Plotnikov & Kichmarenko [96]).

Quasidifferential equations: averaging of quasidifferential equations with fast and slow variables (Mel'nik & Plotnikov [73]); Bogoliubov's first theorem (Mel'nik [71]); partial averaging of quasidifferential equations in metric spaces (Plotnikov & Plotnikova [97]).

Integral differential inclusions: averaging results (Plotnikov & Komleva [81, 88], Janiak & Luczak-Kumorek [47], Vityuk [123], A. Plotnikov [82], Komleva & Arsirii [66]); averaging of integro-quasidifferential equations (Mel'nik [72]).

Topological methods: general fixed point theorem based on averaging principle and its application to differential and integral inclusions (Couchouron & Kamenski [7]).

4 Averaging on an infinite time interval and stability problems

In this section we review the extensions of the averaging method for differential inclusions to an infinite time interval. Although this problem is of utmost importance for the study of nonlinear systems, there are only a few papers devoted to the subject.

The extension of the closeness estimate (7) to the infinite time interval is deeply related to stability proprieties of solutions to the averaged inclusion. Let us recall some basic definitions. Consider a differential inclusion

$$\dot{x} \in G(t, x). \tag{26}$$

Definition 2 A solution $\phi(\cdot) \in \mathcal{S}_{[0,\infty]}(G, \phi_0)$ is called asymptotically stable (weakly asymptotically stable) if given $\epsilon > 0$, there exists $\delta > 0$ such that for any $x_0 \in \phi_0 + \delta B$ each (at least one) solution $x(\cdot) \in \mathcal{S}_{[0,\infty]}(G, x_0)$ satisfies

$|x(t) - \phi(t)| < \epsilon$ for all $t \geq 0$, and

$$\lim_{t\to\infty} |x(t) - \phi(t)| = 0.$$

When $x = 0$ is an equilibrium position of differential inclusion (26), i.e., $0 \in G(t, 0)$ for all $t \geqslant 0$, we have the following definitions.

Definition 3 The equilibrium position $x = 0$ of differential inclusion (26) is said to be asymptotically stable (weakly asymptotically stable) if, given $\epsilon > 0$, there exists $\delta > 0$ such that for any $x_0 \in \delta B$, each (at least one) solution $x(\cdot)$ of (26) with $x(0) = x_0$ satisfies $|x(t)| < \epsilon$ for all $t \geq 0$, and

$$\lim_{t\to\infty} x(t) = 0.$$

Definition 4 We say that the zero equilibrium position of differential inclusion (26) is exponentially stable (weakly exponentially stable), if there exist positive constants c, γ, and δ such that for any $x_0 \in \delta B_n$ all solutions (at least one solution) $x(\cdot)$ of (26) with $x(0) = x_0$ satisfies

$$|x(t)| \leqslant c|x_0|e^{-\gamma t}, \quad t \geq 0.$$

4.1 Stability

The first results relating stability of the averaged inclusion to qualitative proprieties of solutions to the original inclusion, where obtained for Lipschitzian set-valued maps by Plotnikov, see, e.g., [61].

Theorem 5 ([61]) *Assume that the conditions of Theorem 1 hold and besides*
4. the solution $\overline{x}(\cdot) \in \mathcal{S}_{[0,\infty[}(\epsilon\overline{F}, x_0)$ is asymptotically stable.

Then, for any $\eta \in (0, \rho]$ there exists $\epsilon_0 > 0$ and $\sigma > 0$ such that for all $\epsilon \in (0, \epsilon_0]$ and all solutions $x(\cdot) \in \mathcal{S}_{[0,\infty[}(\epsilon F, \tilde{x}_0)$, where $|\tilde{x}_0 - x_0| < \sigma$, we have

$$|x(t) - \overline{x}(t)| < \eta, \quad t > 0.$$

This result, extendes so-called Banfi-Filatov theorem [2,17] to differential inclusions. Similar results for R-solutions can be also found in [61].

In Gama, Guerman & Smirnov [34], it was presented an extension of the Samoilenko-Stanzhitskii theorem [107] to differential inclusions with upper semi-continuous right-hand side. Introducing the set-valued maps

$$G_\epsilon(\tau, y) = F(\tau/\epsilon, y) \text{ and } G_0(y) = \overline{F}(y),$$

it was proved the following theorem.

Theorem 6 ([34]) *Let $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a set-valued map satisfying conditions (C1) - (C5). Assume that $y = 0$ is an asymptotically stable equilibrium position of the differential inclusion $\dot{y} \in G_0(y)$. Then for any $\eta > 0$ there exist $\epsilon_0 > 0$ and $\delta > 0$ such that $\mathcal{S}_{[0,\infty[}(G_\epsilon, \delta B) \subset \eta\mathcal{B}$, whenever $\epsilon \in]0, \epsilon_0[$.*

Later, this result was extended to partially stable differential inclusions in Gama & Smirnov [36].

4.2 Weak stability

For weak stability, a theorem corresponding to Theorem 5 was also proved by Plotnikov for inclusions with Lipschitzian r.h.s., see, e.g., [61].

Theorem 7 ([61]) *Assume that the conditions of Theorem 1 hold and besides*

4. the solution $\overline{x}(\cdot) \in \mathcal{S}_{[0,\infty[}(\epsilon\overline{F}, x_0)$ is weakly asymptotically stable.

Then, for any $\eta \in (0, \rho]$ there exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0]$ there is a solution $x(\cdot) \in \mathcal{S}_{[0,\infty[}(\epsilon F, x_0)$ satisfying

$$|x(t) - \overline{x}(t)| < \eta, \quad t > 0.$$

Recently, Gama & Smirnov, in [38], studied weak exponential stability for time-periodic Lipschitzian differential inclusions. Their approach consists in the application of the averaging method to the first approximation of inclusion (3), allowing the use of easily verifiable sufficient conditions for exponential stability of convex processes from Smirnov [113].

A set-valued map $A : \mathbb{R} \times \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^n)$, with $\mathrm{dom}A(t, \cdot) = \mathbb{R}^n$, $t \geqslant 0$, is a first approximation of the set-valued map $F : \mathbb{R} \times \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^n)$ at the equilibrium position $x = 0$, if $A(t, \cdot)$ is a convex process, i.e. the graph of the map $x \to A(t, x)$ is a closed convex cone for all t, and for any $(x_0, v_0) \in \mathrm{gr}A(t, \cdot)$ and $t \geqslant 0$ the following equality holds:

$$\lim_{h\downarrow 0} h^{-1} d(hv_0, F(t, hx_0)) = 0.$$

If a first approximation is given it is possible to define the differential inclusion of first approximation

$$\dot{x} \in A(t, x). \tag{27}$$

The authors introduced the averaged convex process

$$\overline{A}(\overline{x}) := \left\{ \frac{1}{T} \int_0^T a(s, \overline{x}) ds \mid a(s, \overline{x}) \in A(s, \overline{x}),\ a(s, \overline{x}) \in L_1([0, T], \mathbb{R}^n) \right\},$$

and considered the associated averaged differential inclusion

$$\dot{\overline{x}} \in \epsilon\overline{A}(\overline{x}). \tag{28}$$

Under some natural conditions, they proved the following theorem:

Theorem 8 ([38]) *Let $A : \mathbb{R} \times \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^n)$ be a first approximation of the set-valued map $F : \mathbb{R} \times \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^n)$ at the equilibrium position $x = 0$. Suppose that $\overline{x} = 0$ is a weakly asymptotically stable equilibrium position of inclusion (28). Then, the origin is a weakly exponentially stable equilibrium position of differential inclusion (3), whenever $\epsilon > 0$ is sufficiently small.*

In the same paper, the authors showed that the former result could be applied to differential inclusions generated by control systems. This generalized an earlier work by Gama & Smirnov [37], where weak exponential stability of a linear time-varying control system was studied.

4.3 Related stability problems

Grammel & Maizurna in [44] presented a result concerning uniform exponential stability of time-varying systems with noise. The proof is based on the techniques described in Sec. 3.3. Using the same method, Grammel [42] studied robust exponential stability of systems with delays and singular perturbations. Stability of differential inclusions with multivalued perturbations was studied by Khapaev & Filatov [55] via Lyapunov functions. Filatov in [20] gives a criterion for uniform exponential stability of differential inclusions using comparison method. Klimov in [63] established sufficient conditions for asymptotic stability of equilibrium position of parabolic inclusions.

5 Optimal control and stabilization of differential inclusions

The use of averaging methods in control theory has a long history (see, e.g., the monograph by Akulenko [1]). In this section we review the main trends in the use of averaging method in optimization problems involving differential inclusions.

5.1 Optimal control problems

Let $T = L/\epsilon$. To avoid unnecessary technical details consider the simplest Mayer problem: minimize the functional

$$\phi(x(T)) \to \min \tag{29}$$

over trajectories of the control system

$$\dot{x} \in \epsilon f(t, x, u), \quad u \in U, \quad t \in [0, T], \tag{30}$$

satisfying

$$x(0) = x_0. \tag{31}$$

(Generalization of our considerations to the case with end-point constraints $x(0) \in C_0$ and $x(T) \in C_1$, present no difficulties.) Assume, for simplicity, that the function $t \to f(t, x, u)$ is 2π-periodic.

Note that, even in the case of very simple smooth control systems the averaged ones can be nonsmooth.

Example 4 Consider the control system

$$\dot{x} = \epsilon x u \sin t, \quad \in [-1, 1].$$

The corresponding averaged inclusion is

$$\dot{\overline{x}} \in \epsilon \frac{2}{\pi} |\overline{x}| [-1, 1].$$

Thus, the application of classical optimal control theory [104] to averaged systems is not possible, and generalized differentiation and variational analysis [6,74,106] become essential. Moreover, the use of generalized necessary conditions of optimality, developed to analyze nonsmooth problems, allows one to compare two main averaging techniques: averaging of the original problem and of the two point boundary value problem obtained from Pontryagin's maximum principle. The authors believe that the combination of averaging with the tools of variational analysis is a fruitful approach which will give new interesting results in the near future.

5.1.1 Averaging of the original problem

Under natural conditions control system (30) is equivalent to the differential inclusion

$$\dot{x} \in \epsilon F(t,x), \quad t \in [0,T], \tag{32}$$

where $F(t,x) = f(t,x,U)$. The averaged differential inclusion was first used in the study of optimal control problems by Plotnikov [92]. Although the original Plotnikov's theorem was proved for control systems and the set-valued version of the first Bogolyubov theorem was used only as an auxiliary tool, here we reformulate his result in terms of inclusions. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitzian function. We assume that the r.h.s. of (32) satisfies the conditions of Theorem 1. Consider also the averaged problem: minimize the functional

$$\phi(\overline{x}(T)) \to \min \tag{33}$$

over solutions to the differential inclusion

$$\dot{\overline{x}} \in \epsilon \overline{F}(\overline{x}), \quad t \in [0,T], \tag{34}$$

satisfying

$$\overline{x}(0) = x_0, \tag{35}$$

where $\overline{F}$ is defined by (4). In these terms Plotnikov's result can be formulated as follows.

Theorem 9 *Let $\hat{\overline{x}}(\cdot)$ be a solution to averaged problem (33)-(35). If problem (29), (32), and (31) has a solution $\hat{x}(\cdot)$, then for any $\eta > 0$ and $L > 0$ there exists $\epsilon_0 = \epsilon_0(\eta, L) > 0$ such that for all $\epsilon \in (0, \epsilon_0]$ the inequality*

$$|\phi(\hat{x}(T)) - \phi(\hat{\overline{x}}(T))| < \eta \tag{36}$$

holds.

(Note that the solution to (29), (32), and (31) always exists if F is convex-valued.)

If the class of admissible controls is reduced to the asymptotically constant ones, then a result rather close to Theorem 9 was obtained by Nosenko & Stanzhytskyi [75] using the averaged control system with the r.h.s. defined as

$$\overline{f}(t,x,u) = \frac{1}{2\pi}\int_0^{2\pi} f(t,x,u)dt, \quad u \in U.$$

For a more detailed account of the first results on this topic, we refer the reader to the book by Plotnikov [94]. Some generalizations can be found in A. Plotnikov [79,81,84]. Let us mention some related results. Explicit estimates for the closeness of solutions of the original and averaged optimal control problems were obtained in Dobrodzii [8]. The closeness estimates of the solutions of exact and the corresponding averaged systems were extended to problems with integral functionals on the half-line in Dobrodzii [8] and Stanzhitskii & Dobrodzii [120]. For optimal control problems with impulsive effects some averaging schemes were presented in Kitanov [58]. Quincampoix & Watbled [105] studied a discontinuous Mayer's problem for singularly perturbed control systems.

5.1.2 Pontryagin maximum principle and averaging of the two point boundary value problem

If f and ϕ are smooth enough, the solution to optimal control problem (29)-(31) can be found solving the two point boundary value problem

$$\dot{x} = \epsilon f(t,x,u(t,x,p)), \quad x(0) = x_0, \tag{37}$$

$$\dot{p} = -\epsilon(\nabla_x f(t,x,u(t,x,p)))^* p, \quad p(T) = -\nabla\phi(x(T)), \tag{38}$$

where $u(t,x,p)$ satisfies the Pontryagin maximum principle

$$\max_{u\in U}\langle p, f(t,x,u)\rangle = \langle p, f(t,x,u(t,x,p))\rangle. \tag{39}$$

Another way to treat problem (29)-(31) is to apply the averaging method to system (37) and (38). Set $H(t,x,p) = S(F(t,x),p)$. In terms of Clarke's subdifferentials [6], conditions (37)-(39) can be written as

$$(-\dot{p},\dot{x}) \in \partial_{(x,p)}H(t,x,p), \quad x(0) = x_0, \quad p(T) \in -\partial\phi(x(T)). \tag{40}$$

This form of necessary conditions is applicable also in the case of Lipschitzian F and ϕ. It turns out that, in general, the averaging of inclusion (40) gives weaker conditions of optimality than application of necessary conditions to the averaged problem (33)-(35). Indeed, optimal solution to the averaged problem $\overline{x}(\cdot)$ satisfies the two point boundary value problem

$$(-\dot{\overline{p}},\dot{\overline{x}}) \in \partial_{(\overline{x},\overline{p})}\overline{H}(\overline{x},\overline{p}), \quad \overline{x}(0) = x_0, \quad \overline{p}(T) \in -\partial\phi(\overline{x}(T)), \tag{41}$$

where $\overline{H}(\overline{x},\overline{p}) = S(\overline{F}(\overline{x}),\overline{p})$. Since (see Clarke [6])

$$\partial_{(x,p)}\overline{H}(x,p) = \frac{1}{2\pi}\partial_{(x,p)}S\left(\int_0^{2\pi} F(t,x)dt, p\right) = \frac{1}{2\pi}\partial_{(x,p)}\int_0^{2\pi} S(F(t,x),p)dt$$

$$= \frac{1}{2\pi}\partial_{(x,p)}\int_0^{2\pi} H(t,x,p)dt \subset \frac{1}{2\pi}\int_0^{2\pi} \partial_{(x,p)}H(t,x,p)dt, \quad (42)$$

we see that necessary condition (41) allows one to eliminate from consideration more solution candidates than the averaging of inclusion (40). Moreover, the use of stronger necessary conditions in the form of Euler-Lagrange or partially convexified Hamiltonian inclusions (see Mordukhovich [74]) can made the averaging of the original problem even more efficient tool than the averaging of (40). If F and $\overline{F}$ are sufficiently regular, the inclusion in (42) becomes an equality and both approaches turn out to be equivalent.

5.2 Averaging methods for parameter optimization in discontinuous systems

Consider a differential equation

$$\dot{x} = \epsilon f(t,x,u),\ x \in \mathbb{R}^n, t \geq 0 \quad (43)$$

describing a mechanical system with stabilizer. Here $u \in U \in \mathcal{K}(\mathbb{R}^k)$ is a parameter. It is assumed that $0 \approx f(t,0,u)$ for all $t \geq 0$ and $u \in U$, i.e., that the velocity of the system near the origin is small. Here we don't assume that zero is an equilibrium position of system (43). The parameter $u \in U$ should be chosen to optimize, in some sense, the behavior of the trajectories. The choice of this parameter can be based on various criteria. Obviously, it is impossible to construct a stabilizer optimal in all aspects. Consider, for example, a linear controllable system. The pole assignment theorem guarantees the existence of a linear feedback yielding a linear differential equation with any given set of eigenvalues, so one can choose a stabilizer with a very high damping speed. However, such a stabilizer is practically useless because of so-called peak-effect (see [45,121]). Namely, there exists a large deviation of the solutions from the equilibrium position at the beginning of the stabilization process, whenever the module of the eigenvalues is big.

Motion of many systems is described by differential equations with discontinuous right-hand side. The discontinuity is the principal obstacle in the application of the averaging method. Theorem 6 shows that if the averaged inclusion has zero as its asymptotically stable equilibrium position, the trajectories of the original inclusion stay in the vicinity of the origin provided $\epsilon > 0$ and $|x_0|$ are sufficiently small.

If the averaged inclusion has a special form we can study the behaviour of the trajectories of the original system in detail. Assume that the averaged inclusion has the form

$$\dot{\bar{x}} \in \epsilon(A(u)\bar{x} + \bar{P}(\bar{x},u)), \quad (44)$$

where $\bar{P}(\bar{x},u) \subset c|\bar{x}|^2 B$, $c > 0$, the real parts of the matrix $A(u)$ eigenvalues are negative for all $u \in U$, and the function $u \to A(u)$ is continuous for all $u \in U$. If $\gamma_0 > 0$ is sufficiently small, then the set of solutions to the Lyapunov inequality for the matrix $A(u)$,

$$\mathcal{L}(u) = \{(\gamma, V) \mid \gamma \geq \gamma_0,\ V \in M(n),\ AV + A^*V \leq -2\gamma V\}$$

is nonempty and compact for all $u \in U$. Let $(\gamma, V) \in \mathcal{L}(U)$. Denote by $|\bar{x}|_V$ the Euclidean norm defined by $|\bar{x}|_V = \sqrt{\langle \bar{x}, V\bar{x} \rangle}$. There exist positive constants c_1 and c_2 satisfying

$$c_1|\bar{x}| \leq |\bar{x}|_V \leq c_2|\bar{x}|, \tag{45}$$

whenever $(\gamma, V) \in \mathcal{L}(U)$ for some γ.

Theorem 10 ([35]) *Let $\delta > 0$, $u \in U$, and $(\gamma, V) \in \mathcal{L}(U)$. There exists $\epsilon_0(\delta) > 0$ such that for all $\epsilon \in]0, \epsilon_0(\delta)[$ the condition $|x_0|_V < \delta < c_1^2\gamma/c$ implies the inequality $|x(t, x_0, u)|_V < 3\delta/2$.*

This theorem shows that the behavior of the trajectory $x(t, x_0, u)$ can be characterized in term of the pair (γ, V). The parameter γ is responsible for the damping speed of the process, while the form of the ellipsoid $\{x \mid \langle x, Vx \rangle \leq 1\}$ describes the amplitude of the deviation of the trajectory from the origin. The aim of parameter choosing can be formulated as follows: maximal value of γ and maximal sphericity of the ellipsoid $\{x \mid \langle x, Vx \rangle \leq 1\}$. The latter property guarantees minimal overshooting of the damping process and, as a consequence, the largest region of applicability of the approximation obtained via averaging. In [35] these techniques were used to solve the problem of parameter optimization for a gravitationally stabilized satellite with magnetic hysteresis damper.

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